MULTI-TIME DISTRIBUTION IN DISCRETE POLYNUCLEAR GROWTH

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Abstract. We study the multi-time distribution in a discrete polynuclear growth model or, equivalently, in directed last-passage percolation with geometric weights. A formula for the joint multi-time distribution function is derived in the discrete setting. It takes the form of a multiple contour integral of a block Fredholm determinant. The asymptotic multi-time distribution is then computed by taking the appropriate KPZ-scaling limit of this formula. This distribution is expected to be universal for models in the Kardar-Parisi-Zhang universality class.

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Contents

1. Introduction

Decorate points of $\mathbb{Z}^2$ with independent and identically distributed random weights $\omega(m,n)$ that are non-negative. Associated to this random environment is a growth function $\mathbf{G}$ as follows. For every $m, n \geqslant 1$,

$$\mathbf{G}(m,n) = \max\{\,\mathbf{G}(m-1,n), \mathbf{G}(m,n-1)\,\} + \omega(m,n) \tag{1.1}$$

with boundary conditions $\mathbf{G}(m,0) = \mathbf{G}(0,n) = 0$ for $m, n \geqslant 0$. The function grows out from the corner of the first quadrant along up-right directions, so it is a model of local random growth.

Consider weights chosen according to the geometric law: for some $0 < q < 1$,

$$\mathbf{Pr}\,[\omega(m,n) = k] = (1-q)q^k \quad \text{for } \ k \geqslant 0.$$

The subject of this article is the calculation, and then a derivation of the asymptotic value, of the multi-point probability

$$\mathbf{Pr}\,[\mathbf{G}(m_1,n_1) < a_1, \mathbf{G}(m_2,n_2) < a_2, \ldots, \mathbf{G}(m_p,n_p) < a_p]\,, \tag{1.2}$$

where $m_1 < m_2 < \cdots < m_p$ and $n_1 < n_2 < \cdots < n_p$. In the asymptotic derivation the parameters m, n and a are scaled according to Kardar-Parisi-Zhang (KPZ) scaling [26, 27]. This means that for a large parameter T, the m_ks, n_ks and a_ks are written (ignoring rounding) as

$$\begin{aligned} n_k &= t_k T - c_1 x_k (t_k T)^{\frac{2}{3}}, \\ m_k &= t_k T + c_1 x_k (t_k T)^{\frac{2}{3}}, \\ a_k &= c_2 t_k T + c_3 \xi_k (t_k T)^{\frac{1}{3}}. \end{aligned} \tag{1.3}$$

The c_is are constants that depend on q and will be specified in §2. They are determined from the macroscopic behaviour of $\mathbf{G}(m,n)$. The parameters above are $0 < t_1 < t_2 < \cdots < t_p$, $x_1, x_2, \ldots, x_p \in \mathbb{R}$ and $\xi_1, \xi_2, \ldots, \xi_p \in \mathbb{R}$. One is interested in the large T limit of (1.2) with this scaling.

In Theorem 1 we provide the asymptotic distribution function of $\mathbf{G}$ under KPZ scaling (1.3). Theorem 2 provides an expression for the distribution function (1.2). Theorem 1 is based on an asymptotical analysis of the latter. The calculations leading to Theorem 2, contained in §3 and §4, should be more broadly applicable.

The probability (1.2) is expressed in terms of a $(p-1)$-fold contour integral of a Fredholm determinant involving an $n_p \times n_p$ matrix with a $p \times p$ block structure. This structure persists in the large T limit, and the limiting multi-point probability is expressed by such an integral of some Fredholm determinant over $\mathcal{H} = \underbrace{L^2(\mathbb{R}_{<0}) \oplus \cdots \oplus L^2(\mathbb{R}_{<0})}_{p-1} \oplus L^2(\mathbb{R}_{>0})$.

Interpretation as a growing interface and a non-equilibrium system. The growth model (1.1) has several interpretations. It can be seen as a randomly growing Young diagram, or as a totally asymmetric exclusion process, or yet a directed last passage percolation model, also as a kind of first passage percolation model (with non-positive weights), a system of queues in tandem, and a type of random polymer at zero temperature. A natural interpretation is as a randomly growing interface called discrete polynuclear growth, which we explain.

Rotating the first quadrant 45 degrees, define a function $h(x,t)$ by

$$h(x,t) = \mathbf{G}\left(\frac{t+x+1}{2}, \frac{t-x+1}{2}\right),$$

where $x+t$ is odd, $|x| < t$ and $h(x,0) \equiv 0$. Extend $h(x,t)$ to $x \in \mathbb{Z}$ by linear interpolation. Then (1.1) leads to the rule, see [25], that

$$h(x,t+1) = \max\{\, h(x-1,t), h(x,t), h(x+1,t)\,\} + \eta(x,t+1).$$

The $\eta(x,t)$ are independent and identically distributed with the geometric law if $x+t$ is odd and $|x| < t$, and zero otherwise. This is an instance of the discrete polynuclear growth model, see [28]. If we extend $h(x,t)$ to every $x \in \mathbb{R}$ by linear interpolation then $h(x,t)$ can be thought of as the height above x at time t of a randomly growing interface.

Theorem 1 considers the re-scaled process

$$\mathbf{H}_T(x,t) = \frac{h(2c_1 x (tT)^{\frac{2}{3}},\, 2tT) - c_2 tT}{c_3 (tT)^{\frac{1}{3}}}, \tag{1.4}$$

and provides its joint distribution at the points $(x_1,t_1),\ldots,(x_p,t_p)$ in the large T limit. Since the times are distinct it does not provide all the asymptotic finite dimensional distributions of $\mathbf{H}_T$, although those could be obtained by considering limits in the time parameters. There is in fact a limit function $\mathbf{H}(x,t)$ that is continuous almost surely, see [30], which means that in principle the aforementioned distributions do determine the law of $\mathbf{H}$. As can be seen from (1.3) and (1.4), we study time-like distributions of $\mathbf{H}_T$ in the $(1,1)$ direction of the (m,n)-plane. In other directions we expect the distributions to become asymptotically independent since non-trivial spatial correlations only occur at a scale of $T^{2/3}$. Therefore we look in the so called characteristic direction; see [16] for further discussion on this.

By re-scaling variables in the kernel from Theorem 1 it may be seen that for every $\lambda > 0$, $\mathbf{H}(x,\lambda t)$ has the same distribution as $\mathbf{H}(x,t)$ as functions of x and t. If we define $\mathbf{A}(x,t) = t^{1/3}\mathbf{H}(t^{-2/3}x,t) + t^{-1}x^2$, then this means that

$$\lambda^{-\frac{1}{3}} \cdot \mathbf{A}(\lambda^{\frac{2}{3}} x,\, \lambda t) \overset{law}{=} \mathbf{A}(x,t).$$

The relation above is known as KPZ scale invariance which, in this context, makes the polynuclear growth model a part of the KPZ universality class. The latter is a collection of 1+1 dimensional statistical mechanical systems whose fluctuations demonstrate the scale invariance above. Within the KPZ universality class lies the Airy$_2$ process (see [9, 25, 32] for reference), which represents asymptotic spatial fluctuations in x of the height function at a fixed time t. So $\mathbf{A}(x,t)$ may be thought of as the space-time surface sketched out by a growing Airy interface. Some surveys that discuss these topics in depth are [5, 7, 18, 33], and [38] is a nice introduction to the growth model.

The papers [1, 6, 10, 19, 30] have recently studied various aspects of limit distributions in the KPZ universality class. Here we find for the first time a full multi-time distribution function in the KPZ-scaling limit. A multi-time distribution function is actually derived in [1] for the related continuous time TASEP in a periodic setting, and the asymptotic limit is computed in the relaxation time-scale, when the TASEP is affected by the finite geometry. It is not obvious how to get the asymptotic result of the present paper from theirs, since it means computing asymptotics in a situation where the TASEP is not affected by the finite geometry. However, after the completion of this work, the paper [29] derived the multi-time distribution for the continuous time TASEP in the infinite geometry. The relation between the formulas before the limit in [1, 29] and the one in this paper is not clear so far.

The present paper generalizes previous work on the two-time distribution in [22]. The two-time distribution has also been investigated in the theoretical physics literature, see [11, 12, 13] and references there. Moreover, correlation function of the two-time distribution has been studied in [2, 17]. The distribution of this growth model under a different asymptotic scaling, related to the slow decorrelation phenomenon, has also been explored in [4, 8, 16, 20]. Finally, see the paper [35] for some nice experimental work involving growth interfaces in liquid crystal.

Additional remarks. The formula for the limiting distribution function for $\mathbf{H}(x,t)$ in Theorem 1 is rather complicated. It is built from kernels given by compositions of Airy functions, which thus generalizes the Airy kernel. In the two-time case it is possible to rewrite the formula in such a way that the limits $t_2/t_1 \to 1$ and $t_2/t_1 \to \infty$ may be studied in detail, see [23]. It would be interesting to do the same for the Fredholm determinant in Theorem 1, so that these types of limits can be analyzed in the multi-time case as well. The distribution can in fact be computed numerically starting from the formula in Theorem 1 in the two-time case, see [14], which shows that although complicated the formula is useful nonetheless.

In this paper we study the case of geometrically distributed weights $\omega(m,n)$. The case of exponentially distributed weights can be obtained by taking the appropriate limit ($q \to 1$) in the discrete formula. Similarly, the Brownian directed polymer model can be obtained as a limit. The asymptotic analysis is completely analogous. We expect the limiting multi-time formula in Theorem 1 to be universal within a large class of models. It should be possible to study the limit of Poissonian last-passage percolation (Poissonized Plancherel) ($q \to 0$) from our formula in Theorem 2, but this would entail taking a limit to an infinite Fredholm determinant before the large time asymptotics are computed.

2. Statement of results

In order to state the theorems we have to introduce notation. There is quite a bit of notation throughout the article, so in the following, we introduce notation for both statement of theorems and ones that recur.

2.1. **Some notation and conventions.** Consider times $0 < t_1 < t_2 < \cdots < t_p$, points $x_1, x_2, \dots x_p \in \mathbb{R}$ and $\xi_1, \xi_2, \dots, \xi_p \in \mathbb{R}$. Introduce the scaling constants

$$c_0 = q^{-\frac{1}{3}}(1+\sqrt{q})^{\frac{1}{3}}, \quad c_1 = q^{-\frac{1}{6}}(1+\sqrt{q})^{\frac{2}{3}}, \quad c_2 = \frac{2\sqrt{q}}{1-\sqrt{q}}, \quad c_3 = \frac{q^{\frac{1}{6}}(1+\sqrt{q})^{\frac{1}{3}}}{1-\sqrt{q}}, \tag{2.1}$$

where q is the parameter of the geometric distribution. We will investigate the asymptotics of the probability distribution given by (1.2) under the scaling (1.3).

Delta notation. For integers $0 \leqslant k_1 < k_2 \leqslant p$, and y being m, n or a from (1.3), define

$$\Delta_{k_1,k_2} y = y_{k_2} - y_{k_1} \quad \text{and} \quad \Delta_k y = y_k - y_{k-1}. \tag{2.2}$$

Also, define

$$\begin{aligned} &\Delta_{k_1,k_2} t = t_{k_2} - t_{k_1} \quad \text{and} \quad \Delta_k t = t_k - t_{k-1}, \\ &\Delta_{k_1,k_2} x = x_{k_2} \left(\frac{t_{k_2}}{\Delta_{k_1,k_2} t}\right)^{\frac{2}{3}} - x_{k_1} \left(\frac{t_{k_1}}{\Delta_{k_1,k_2} t}\right)^{\frac{2}{3}} \quad \text{and} \quad \Delta_k x = \Delta_{k-1,k} x\,, \\ &\Delta_{k_1,k_2} \xi = \xi_{k_2} \left(\frac{t_{k_2}}{\Delta_{k_1,k_2} t}\right)^{\frac{1}{3}} - \xi_{k_1} \left(\frac{t_{k_1}}{\Delta_{k_1,k_2} t}\right)^{\frac{1}{3}} \quad \text{and} \quad \Delta_k \xi = \Delta_{k-1,k} \xi\,. \end{aligned} \tag{2.3}$$

By convention, $y_0 = 0$ for $y = n, m, a, t, x, \xi$. To understand (2.3) note that it is such that $\Delta_{k_1,k_2} n = (\Delta_{k_1,k_2} t)T - c_1 \Delta_{k_1,k_2} x\,(\Delta_{k_1,k_2} t\,T)^{\frac{2}{3}}$, and similarly for the differences between m_ks and a_ks. We will also use the shorthand

$$\Delta_{k_1,k_2}(y^1, \ldots, y^\ell) = (\Delta_{k_1,k_2} y^1, \ldots, \Delta_{k_1,k_2} y^\ell) \quad \text{and} \quad \Delta_k(y^1, \ldots, y^\ell) = (\Delta_k y^1, \ldots, \Delta_k y^\ell).$$

Block notation. The matrices that appear will have a $p \times p$ block structure with the rows and columns partitioned according to

$$\{1, 2, \ldots, n_p\} = (0, n_1] \cup (n_1, n_2] \cup \cdots \cup (n_{p-1}, n_p].$$

The following notation will help us with calculations that depend on this structure. For $y = m, n, a$, set

$$\begin{aligned} y(k) &= y_{\min\{r, p-1\}} \quad \text{if} \;\; k \in (n_{r-1}, n_r], \\ r^* &= \min\{r, p-1\} \quad \text{if} \;\; 1 \leqslant r \leqslant p. \end{aligned} \tag{2.4}$$

For an $n_p \times n_p$ matrix M, $1 \leqslant i, j \leqslant n_p$ and $1 \leqslant r, s \leqslant p$, write

$$M(r, i; s, j) = \mathbf{1}_{\left\{i \in (n_{r-1}, n_r],\, j \in (n_{s-1}, n_s]\right\}} \cdot M(i, j).$$

This is the $p \times p$ block structure of M according to the partition of rows and columns above.

Suppose $1 \leqslant i \leqslant n_p$. For $\vec{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_{p-1}) \in \{1, 2\}^{p-1}$ and $\theta = (\theta_1, \ldots, \theta_{p-1}) \in (\mathbb{C} \setminus 0)^{p-1}$, define the following quantities.

$$\begin{aligned} \theta^{\vec{\varepsilon}}(i) &= \prod_{k=1}^{p-1} \theta_k^{2 - \varepsilon_k - \mathbf{1}_{\{i \leqslant n_k\}}}, \\ \theta(r \,|\, \vec{\varepsilon}) &= \prod_{k=1}^{r-1} \theta_k^{2 - \varepsilon_k} \prod_{k=r}^{p-1} \theta_k^{1 - \varepsilon_k} \quad \text{for} \;\; 1 \leqslant r \leqslant p. \end{aligned} \tag{2.5}$$

Observe that $\theta^{\vec{\varepsilon}}(i) = \theta(r \,|\, \vec{\varepsilon})$ for every $i \in (n_{r-1}, n_r]$, so these are block functions. Particularly notable $\vec{\varepsilon}$ will be

$$\varepsilon^k = (\overbrace{2, \ldots, 2}^{k-1}, 1, \ldots, 1) \quad \text{for} \;\; 1 \leqslant k \leqslant p.$$

For these we define

$$\Theta(r \mid k) = \theta(r \mid \varepsilon^k) - (1 - \mathbf{1}_{\{r=p,k=p-2\}}) \cdot \theta(r \mid \varepsilon^{k+1}) \quad \text{for } 1 \leqslant k < \min\{r, p-1\},\ 1 \leqslant r \leqslant p. \tag{2.6}$$

We may set $\Theta(r \mid k)$ to be zero otherwise. Let us also set

$$(-1)^{\varepsilon[k_1,k_2]} = (-1)^{\sum_{k=\max\{1,k_1\}}^{\min\{k_2,p-1\}} \varepsilon_k} \quad \text{for } 0 \leqslant k_1 < k_2 \leqslant p.$$

It will be convenient to write $(-1)^{\varepsilon[k_1,k_2]} \cdot (-1)^x$ as $(-1)^{\varepsilon[k_1,k_2]+x}$.

Define also the indicators functions

$$\chi_\varepsilon(x) = \begin{cases} \mathbf{1}_{\{x<0\}} & \text{if } \varepsilon \equiv 1 \mod 2, \\ \mathbf{1}_{\{x \geqslant 0\}} & \text{if } \varepsilon \equiv 2 \mod 2. \end{cases} \tag{2.7}$$

Complex integrands. Define, for $n, m, a \in \mathbb{Z}$ and $w \in \mathbb{C} \setminus \{0, 1-q, 1\}$,

$$G^*(w \mid n, m, a) = \frac{w^n (1-w)^{a+m}}{\left(1 - \frac{w}{1-q}\right)^m}, \tag{2.8}$$

as well as the function

$$G(w \mid n, m, a) = \frac{G^*(w \mid n, m, a)}{G^*(1-\sqrt{q} \mid n, m, a)}. \tag{2.9}$$

The number $w_c = 1 - \sqrt{q}$ is the critical point around which we will perform steepest descent analysis. During the asymptotical analysis it will be convenient to write in terms of G rather than G^*. Consider also the following function $\mathcal{G}$ that will become the asymptotical value of G.

$$\mathcal{G}(w \mid t, x, \xi) = \exp\left\{ \frac{t}{3} w^3 + t^{\frac{2}{3}} x w^2 - t^{\frac{1}{3}} \xi w \right\} \quad \text{for } w \in \mathbb{C} \text{ and } t, x, \xi \in \mathbb{R}. \tag{2.10}$$

Contour notation. We will always denote the contour integral

$$\frac{1}{2\pi \mathbf{i}} \int_\gamma dz \quad \text{as} \quad \oint_\gamma dz\,.$$

There will be two types of contours in our calculations: circles and vertical lines. Throughout, γ_r denotes a circular contour around the origin of radius $r > 0$ with counterclockwise orientation. Also, $\gamma_r(1)$ is such a circular contour around 1. A vertical contour through $d \in \mathbb{R}$ oriented upwards is denoted Γ_d.

Conjugations. Throughout the article μ will denote a sufficiently large constant used with a conjugation factor. Its value will depend only on the parameters q, t_k, x_k and ξ_k. It will be convenient to not state the value of μ explicitly, although in the upcoming theorem it suffices to consider

$$\mu > \frac{\max_k \{x_k t_k^{2/3}\} - \min_k \{x_k t_k^{2/3}\}}{\min_k \{(\Delta_k t)^{1/3}\}}\,.$$

Define, with c_0 given by (2.1),

$$\nu_T = c_0 T^{1/3}.$$

Let us introduce discrete conjugation factors, which will be needed for asymptotical analysis. Recall $n(k), m(k)$ and $a(k)$ from (2.4). For $1 \leqslant k \leqslant n_p$,

$$c(k) = G^*\big(1-\sqrt{q}\,|\,k, m(k), a(k)\big)\cdot e^{\mu\frac{(n(k)-k)}{\nu_T}}. \tag{2.11}$$

Finally, set

$$c(i,j) = \exp\Big\{\mu\frac{(n(i)-i)-(n(j)-j)}{\nu_T}\Big\}. \tag{2.12}$$

2.2. **Statement of main theorem.** For $p \geqslant 1$ consider the Hilbert space

$$\mathcal{H} = \underbrace{L^2(\mathbb{R}_{<0})\oplus\cdots\oplus L^2(\mathbb{R}_{<0})}_{p-1}\oplus L^2(\mathbb{R}_{>0}).$$

A kernel F on $\mathcal{H}$ has a $p\times p$ block structure, and we denote by $F(r,u;s,v)$ its (r,s)-block. So

$$F(u,v) = \begin{bmatrix} F(1,u;1,v) & \cdots & F(1,u;p,v) \\ \vdots & & \vdots \\ F(p,u;1,v) & \cdots & F(p,u;p,v)\end{bmatrix}_{p\times p}.$$

Recall the function $\mathcal{G}$ from (2.10), the notation $r^* = \min\{r, p-1\}$ and likewise s^* from (2.4).

Definition 2.1. The following basic matrix kernels over $\mathcal{H}$ will constitute a final kernel.

(1) Let $d_1 > 0$ and $D > 0$. Define

$$F[p\,|\,p]\,(r,u;s,v) = \mathbf{1}_{\{r=p\}}\, e^{\mu(v-u)} \oint_{\Gamma_{-d_1}} d\zeta_1 \oint_{\Gamma_D} dz_p\, \frac{\mathcal{G}\big(z_p\,|\,\Delta_p(t,x,\xi)\big)\, e^{\zeta_1 v - z_p u}}{\mathcal{G}\big(\zeta_1\,|\,\Delta_{s^*,p}(t,x,\xi)\big)\,(z_p-\zeta_1)}.$$

Recall Γ_d is a vertical contour oriented upwards that intersects the real axis at d.

(2) Let $0 < d_1 < d_2$. For $0 \leqslant k \leqslant p$, define

$$F[k,k\,|\,\emptyset]\,(r,u;s,v) = \mathbf{1}_{\{s<k<r^*\}}\, e^{\mu(v-u)} \oint_{\Gamma_{-d_1}} d\zeta_1 \oint_{\Gamma_{-d_2}} d\zeta_2$$
$$\frac{(\zeta_1-\zeta_2)^{-1}\, e^{\zeta_2 v - \zeta_1 u}}{\mathcal{G}\big(\zeta_1\,|\,\Delta_{k,r^*}(t,x,\xi)\big)\,\mathcal{G}\big(\zeta_2\,|\,\Delta_{s,k}(t,x,\xi)\big)}.$$

(3) Let $0 < d_3 < d_2$ and $D > 0$. For $0 \leqslant k \leqslant p$, define

$$F[p,k\,|\,p]\,(r,u;s,v) = \mathbf{1}_{\{r=p,\,s<k<p\}}\, e^{\mu(v-u)} \oint_{\Gamma_{-d_2}} d\zeta_2 \oint_{\Gamma_{-d_3}} d\zeta_3 \oint_{\Gamma_D} dz_p$$
$$\frac{\mathcal{G}\big(z_p\,|\,\Delta_p(t,x,\xi)\big)(z_p-\zeta_2)^{-1}(\zeta_2-\zeta_3)^{-1} e^{\zeta_3 v - z_p u}}{\mathcal{G}\big(\zeta_2\,|\,\Delta_{k,p}(t,x,\xi)\big)\mathcal{G}\big(\zeta_3\,|\,\Delta_{s,k}(t,x,\xi)\big)}.$$

(4) Let $0 < d_1, d_3 < d_2$. For $0 \leqslant k_1, k_2 \leqslant p$, define

$$F[k_1,k_1,k_2 \,|\, \emptyset]\,(r,u;s,v) = \mathbf{1}_{\{k_1<r^*,\, s<k_2<k_1\}}\, e^{\mu(v-u)} \oint_{\Gamma_{-d_1}} d\zeta_1 \oint_{\Gamma_{-d_2}} d\zeta_2 \oint_{\Gamma_{-d_3}} d\zeta_3$$
$$\frac{(\zeta_1-\zeta_2)^{-1}\,(\zeta_2-\zeta_3)^{-1}\,e^{\zeta_3 v-\zeta_1 u}}{\mathcal{G}\big(\zeta_1 \,|\, \Delta_{k_1,r^*}(t,x,\xi)\big)\,\mathcal{G}\big(\zeta_2 \,|\, \Delta_{k_2,k_1}(t,x,\xi)\big)\,\mathcal{G}\big(\zeta_3 \,|\, \Delta_{s,k_2}(t,x,\xi)\big)}.$$

The upcoming kernels are determined in terms of integer parameters $0 \leqslant k_1 < k_2 \leqslant p$ and a vector parameter $\vec{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_{p-1}) \in \{1,2\}^{p-1}$. Given k_1, k_2 and $\vec{\varepsilon}$, consider any set of distinct positive real numbers D_k for integers $k \in (k_1, k_2]$ that satisfy the following pairwise ordering:

$$D_k < D_{k+1} \text{ if } \varepsilon_k = 1 \text{ while } D_k > D_{k+1} \text{ if } \varepsilon_k = 2. \tag{2.13}$$

It is easy to see, for instance by induction, that it is always possible to order distinct real numbers such that they satisfy these constraints imposed by $\vec{\varepsilon}$. An explicit choice would be

$$D_1 = 2^p \quad \text{and} \quad D_{k+1} = D_k + (-1)^{\varepsilon_k+1}\, 2^k.$$

Denote the contour

$$\vec{\Gamma}_{D^{\vec{\varepsilon}}} = \Gamma_{D_{k_1+1}} \times \cdots \times \Gamma_{D_{k_2}}.$$

(5) Let $d_1 > 0$. Define

$$F^{\vec{\varepsilon}}[k_1 \,|\, (k_1,k_2]]\,(r,u;s,v) = \mathbf{1}_{\{k_1<r^*,\, s=k_2<p,\, k_1<k_2\}}\, e^{\mu(v-u)} \oint_{\Gamma_{-d_1}} d\zeta_1 \oint_{\vec{\Gamma}_{D^{\vec{\varepsilon}}}} dz_{k_1+1}\cdots dz_{k_2}$$
$$\frac{\prod_{k_1<k\leqslant k_2} \mathcal{G}\big(z_k \,|\, \Delta_k(t,x,\xi)\big)\, \prod_{k_1<k<k_2} (z_k - z_{k+1})^{-1}\, e^{z_{k_2} v - \zeta_1 u}}{\mathcal{G}\big(\zeta_1 \,|\, \Delta_{k_1,r^*}(t,x,\xi)\big)(z_{k_1+1}-\zeta_1)}.$$

(6) Let $d_1, d_2 > 0$. Define

$$F^{\vec{\varepsilon}}[k_1,k_2 \,|\, (k_1,k_2]]\,(r,u;s,v) = \mathbf{1}_{\{k_1<r^*,\, s^*<k_2,\, k_1<k_2\}}\, e^{\mu(v-u)} \oint_{\Gamma_{-d_1}} d\zeta_1 \oint_{\Gamma_{-d_2}} d\zeta_2 \oint_{\vec{\Gamma}_{D^{\vec{\varepsilon}}}} dz_{k_1+1}\cdots dz_{k_2}$$
$$\frac{\prod_{k_1<k\leqslant k_2} \mathcal{G}\big(z_k \,|\, \Delta_k(t,x,\xi)\big)\, \prod_{k_1<k<k_2} (z_k - z_{k+1})^{-1}\, e^{\zeta_2 v - \zeta_1 u}}{\mathcal{G}\big(\zeta_1 \,|\, \Delta_{k_1,r^*}(t,x,\xi)\big)\,\mathcal{G}\big(\zeta_2 \,|\, \Delta_{s^*,k_2}(t,x,\xi)\big)\,(z_{k_1+1}-\zeta_1)\,(z_{k_2}-\zeta_2)}.$$

(7) Let $0 < d_1, d_3 < d_2$ and recall $k_1 < k_2$. Define

$$F^{\vec{\varepsilon}}[k_1,k_2,k_3 \,|\, (k_1,k_2]](r,u;s,v) = \mathbf{1}_{\{k_1<r^*,\, s<k_3<k_2\}} e^{\mu(v-u)} \oint_{\Gamma_{-d_1}} d\zeta_1 \oint_{\Gamma_{-d_2}} d\zeta_2 \oint_{\Gamma_{-d_3}} d\zeta_3 \oint_{\vec{\Gamma}_{D^{\vec{\varepsilon}}}} dz_{k_1+1}\cdots dz_{k_2}$$
$$\frac{\prod_{k_1<k\leqslant k_2} \mathcal{G}\big(z_k \,|\, \Delta_k(t,x,\xi)\big)\, \prod_{k_1<k<k_2} (z_k - z_{k+1})^{-1}\,(\zeta_2-\zeta_3)^{-1}\, e^{\zeta_3 v - \zeta_1 u}}{\mathcal{G}\big(\zeta_1 \,|\, \Delta_{k_1,r^*}(t,x,\xi)\big)\,\mathcal{G}\big(\zeta_2 \,|\, \Delta_{k_3,k_2}(t,x,\xi)\big)\,\mathcal{G}\big(\zeta_3 \,|\, \Delta_{s,k_3}(t,x,\xi)\big)(z_{k_1+1}-\zeta_1)\,(z_{k_2}-\zeta_2)}.$$

When the conjugation constant μ is sufficiently large these kernels decay rapidly to be of trace class, which will be a byproduct of the proof of Theorem 1. (Specifically, their entries are bounded by quantities of the form $e^{-\tilde{\mu}u}\mathrm{Ai}(-u)\, e^{\tilde{\mu}v}\mathrm{Ai}(v)$ where Ai is the Airy function.)

Using these basic kernels we compose five other as weighted sums. Let $\theta_1, \ldots, \theta_{p-1}$ be non-zero complex numbers and $\theta = (\theta_1, \ldots, \theta_{p-1})$. Recall $\theta(r \mid \vec{\varepsilon})$ and $\Theta(r \mid k)$ from (2.5) and (2.6), respectively. Define the following kernels over $\mathcal{H}$.

$$F^{(0)}(r,u;s,v) = \sum_{0 \leqslant k \leqslant p} (1 + \Theta(r \mid k)) \cdot (1 + \Theta(k \mid s)) \cdot F[k,k \mid \emptyset]\,(r,u;s,v).$$

$$F^{(1)}(r,u;s,v) = \sum_{0 \leqslant k \leqslant p} \Theta(r \mid k) \cdot F[k,k \mid \emptyset]\,(r,u;s,v).$$

$$F^{(3)}(r,u;s,v) = \sum_{0 \leqslant k_1,k_2 \leqslant p} \Theta(r \mid k_1) \cdot \big(1 + \Theta(k_2 \mid s)\big) \cdot F[k_1,k_1,k_2 \mid \emptyset]\,(r,u;s,v).$$

In the following, the variables $k_1, k_2, k_3 \in \{0, \ldots, p\}$ and $\vec{\varepsilon} \in \{1,2\}^{p-1}$. They satisfy

$$k_1 < k_2; \quad \text{given } k_1, k_2,\ \vec{\varepsilon} = (\overbrace{2, \ldots, 2}^{\substack{\varepsilon_i = 2 \text{ if} \\ i < \max\{k_1,1\}}}, \overbrace{\varepsilon_{\max\{k_1,1\}}, \ldots, \varepsilon_{\min\{k_2,p-1\}}}^{\text{arbitrary 1 or 2}}, \overbrace{1, \ldots, 1}^{\substack{\varepsilon_i = 1 \text{ if} \\ i > \min\{k_2,p-1\}}}). \tag{2.14}$$

Recall the notation $(-1)^{\varepsilon[k_1,k_2]}$ following (2.6). Define

$$F^{(2)}(r,u;s,v) = \sum_{\substack{k_1,k_2,\vec{\varepsilon} \\ \text{satisfies (2.14)}}} (-1)^{\varepsilon[k_1,k_2] + \mathbf{1}_{\{k_2=p\}}} \cdot \theta(r \mid \vec{\varepsilon}) \times$$
$$\Big[F^{\vec{\varepsilon}}[k_1 \mid (k_1,k_2)] + F^{\vec{\varepsilon}}[k_1,k_2 \mid (k_1,k_2)] + \mathbf{1}_{\{k_1=p-1,\,k_2=p\}} F[p \mid p]\Big](r,u;s,v).$$

$$F^{(4)}(r,u;s,v) = \sum_{\substack{k_1,k_2,k_3,\vec{\varepsilon} \\ \text{satisfies (2.14)}}} (-1)^{\varepsilon[k_1,k_2] + \mathbf{1}_{\{k_2=p\}}} \cdot \theta(r \mid \vec{\varepsilon}) \times$$
$$\Big[\big(1 + \Theta(k_3 \mid s)\big) F^{\vec{\varepsilon}}[k_1,k_2,k_3 \mid (k_1,k_2)] - \mathbf{1}_{\{k_2=p,\,k_3=p-1\}}\big(1 + \Theta(p \mid s)\big) F^{\vec{\varepsilon}}[k_1,p,p-1 \mid (k_1,p)] +$$
$$\mathbf{1}_{\{k_2<p,\,k_3=p\}}\big(1 + \Theta(k_2 \mid s)\big) F^{\vec{\varepsilon}}[k_1,k_2 \mid (k_1,k_2)] +$$
$$\mathbf{1}_{\{k_1=p-1,k_2=p\}}\big(1 + \Theta(k_3 \mid s)\big) F[p,k_3 \mid p] - \mathbf{1}_{\{k_1=p-1,k_2=p,k_3=p-1\}}\big(1 + \Theta(p \mid s)\big) F[p,p-1 \mid p]\big)\Big](r,u;s,v).$$

Finally, define the kernel

$$F(\theta) = -F^{(0)} + F^{(1)} + F^{(2)} - F^{(3)} - F^{(4)}. \tag{2.15}$$

Theorem 1. *Consider the function* $\mathbf{G}(m,n)$ *from* (1.1)*. Let* n_k, m_k *and* a_k *be scaled according to* (1.3) *with respect to parameters* T, t_k, x_k *and* ξ_k*. Suppose* $p \geqslant 2$*. Then,*

$$\lim_{T \to \infty} \mathbf{Pr}\left[\mathbf{G}(m_1,n_1) < a_1, \ldots, \mathbf{G}(m_p,n_p) < a_p\right] =$$
$$\oint_{\gamma_r} d\theta_1 \cdots \oint_{\gamma_r} d\theta_{p-1} \frac{\det\left(I + F(\theta)\right)_{\mathcal{H}}}{\prod_k (\theta_k - 1)}$$

where γ_r *is a counter-clockwise circular contour around the origin of radius* $r > 1$ *and* $F(\theta)$ *is from* (2.15)*. Moreover, the limit defines a consistent family of probability distribution functions.*

When $p = 2$ this theorem agrees with the two-time distribution function from [22]. In this case the only non-zero component of $F(\theta)$ is $F^{(2)}$, whose non-zero basic kernels are $F[0 \mid (0,1)]$, $F[2 \mid 2]$ and

$F^{\varepsilon}[0,2\,|\,(0,2]]$ for $\varepsilon = 1,2$. Our other theorem that presents a similar expression for the probability (1.2) is stated as Theorem 2, towards the end of §4.

2.3. **A discussion of results.**

Single point law. When $p = 1$ there is a simpler approach for the single point limit as explained in §4.3, where we express $\mathbf{Pr}[\mathbf{G}(m,n) < a]$ as a Fredholm determinant of a matrix whose entries are in terms of a double contour integral. More precisely, $\mathbf{Pr}[\mathbf{G}(m,n) < a] = \det(I + M)$ with

$$M(i,j) = \oint_{\gamma_\tau} d\zeta \oint_{\gamma_r(1)} dz \, \frac{G^*(z\,|\,n-i, m, a-1)}{G^*(\zeta\,|\,n-j+1, m, a-1)(z-\zeta)}.$$

Here $1 \leqslant i,j \leqslant n$ and the radii satisfy $\tau < 1-\sqrt{q} < 1-r < 1-q$.

An asymptotical analysis of it leads to

$$\lim_{T\to\infty} \mathbf{Pr}[\mathbf{G}(m_1,n_1) < a_1] = \det(I-K)_{L^2(\mathbb{R}_{>0})}, \text{ where} \tag{2.16}$$

$$K(u,v) = \oint_{\Gamma_{-d}} d\zeta \oint_{\Gamma_D} dz \, \frac{\mathcal{G}(z\,|\,t_1,x_1,\xi_1)}{\mathcal{G}(\zeta\,|\,t_1,x_1,\xi_1)} \cdot \frac{e^{\zeta v - zu}}{z-\zeta}.$$

One may observe that

$$\oint_{\Gamma_D} dz \, \mathcal{G}(z\,|\,t,x,\xi) e^{-zu} = t^{-\frac{1}{3}} \, e^{\frac{2}{3}x^3 + (\xi + t^{-\frac{1}{3}}u)x} \, \mathrm{Ai}(\xi + x^2 + t^{-\frac{1}{3}}u). \tag{2.17}$$

Using this, as well as $\oint_{\Gamma_{-d}} d\zeta \, \mathcal{G}(\zeta\,|\,t,x,\xi)^{-1} e^{\zeta v} = \oint_{\Gamma_d} d\zeta \, \mathcal{G}(\zeta\,|\,t,-x,\xi) e^{-\zeta v}$, and that $(z-\zeta)^{-1} = \int_0^\infty d\lambda \, e^{\lambda(\zeta - z)}$, we find that

$$e^{x(v-u)} t^{\frac{1}{3}} K(t^{\frac{1}{3}}u, t^{\frac{1}{3}}v) = \int_0^\infty d\lambda \, \mathrm{Ai}(\xi + x^2 + u + \lambda)\mathrm{Ai}(\xi + x^2 + v + \lambda) = K_{\mathrm{Ai}}(\xi + x^2 + u, \xi + x^2 + v).$$

This implies that $\det(I-K)_{L^2(\mathbb{R}_{>0})}$ equals $F_{\mathrm{GUE}}(\xi + x^2)$, where F_{GUE} is the distribution function of the GUE Tracy-Widom law from [39]. The single point law recovers a result from [24].

Kernels expressed in terms of Airy function. The kernels in Definition 2.1 may be written as products of more basic ones. Consider the following kernel for $x, \xi \in \mathbb{R}$ and $t > 0$:

$$\mathcal{A}[t,x,\xi](u,v) = \oint_{\Gamma_D} dw \, \mathcal{G}(w\,|\,t,x,\xi) e^{w(u-v)} = t^{-\frac{1}{3}} \mathrm{Ai}\big(x^2 + \xi + t^{-\frac{1}{3}}(v-u)\big) \, e^{\frac{2}{3}x^2 + x(\xi + t^{-\frac{1}{3}}(v-u))}.$$

We will show how to write $F^{\vec{\varepsilon}}[k_1,k_2\,|\,(k_1,k_2]]$ using $\mathcal{A}$ and the others are done similarly. Observe $(w_1 - w_2)^{-1} = \int_0^\infty d\lambda \, e^{-\lambda(w_1-w_2)\cdot \mathrm{sgn}(\Re(w_1-w_2))}$. As a result,

$$(z_k - z_{k+1})^{-1} = \int_0^\infty d\lambda_k \, e^{\lambda_k (-1)^{\varepsilon_k} (z_{k+1} - z_k)} \quad \text{for} \;\; k_1 < k < k_2,$$

$$(z_{k_1+1} - \zeta_1)^{-1} = \int_0^\infty d\lambda_{k_1} e^{\lambda_{k_1}(\zeta_1 - z_{k_1+1})}, \;\; (z_{k_2} - \zeta_2)^{-1} = \int_0^\infty d\lambda_{k_2} \, e^{\lambda_{k_2}(\zeta_2 - z_{k_2})}.$$

Let us set $\varepsilon_{k_1} = 1$ and $\varepsilon_{k_2} = 2$ in the following. Then we see that

$$F^{\vec{\varepsilon}}[k_1,k_2\,|\,(k_1,k_2]](r,u;s,v) = \mathbf{1}_{\{k_1<r^*,\, s^*<k_2,\, k_1<k_2\}}\, e^{\mu(v-u)} \int_{[0,\infty)^{[k_1,k_2]}} \prod_{k_1\leqslant k\leqslant k_2} d\lambda_k$$
$$\oint_{\Gamma_{-d_1}} d\zeta_1\, \mathcal{G}\big(\zeta_1\,|\,\Delta_{k_1,r^*}(t,x,\xi)\big)^{-1} e^{\zeta_1(\lambda_{k_1}-u)} \oint_{\Gamma_{-d_2}} d\zeta_2\, \mathcal{G}\big(\zeta_2\,|\,\Delta_{s^*,k_2}(t,x,\xi)\big)^{-1} e^{\zeta_2(\lambda_{k_2}+v)}$$
$$\prod_{k_1<k\leqslant k_2} \oint_{\Gamma_{D_k}} dz_k\, \mathcal{G}\big(z_k\,|\,\Delta_k(t,x,\xi)\big)\, e^{z_k[(-1)^{\varepsilon_{k-1}}\cdot\lambda_{k-1}-(-1)^{\varepsilon_k}\cdot\lambda_k]}.$$

We can evaluate the ζ-integrals by changing variables $\zeta \to -\zeta$ as in the single time discussion. Let us consider also the reflection R for which $R\cdot K(u,v) = K(-u,v)$. We have $K((-1)^{\varepsilon}u,(-1)^{\varepsilon'}v) = R^{\varepsilon}KR^{\varepsilon'}(u,v)$. Then we find that

$$F^{\vec{\varepsilon}}[k_1,k_2\,|\,(k_1,k_2]](r,u;s,v) = \mathbf{1}_{\{k_1<r^*,\, s^*<k_2,\, k_1<k_2\}}\, e^{\mu(v-u)} \int_{[0,\infty)^{]k_1,k_2]}} \prod_{k_1\leqslant k\leqslant k_2} d\lambda_k$$
$$\mathcal{A}[\Delta_{k_1,r^*}t,\, -\Delta_{k_1,r^*}x,\, \Delta_{k_1,r^*}\xi]\,(u,\lambda_{k_1}) \prod_{k_1<k\leqslant k_2} R^{\varepsilon_{k-1}}\mathcal{A}[\Delta_k(t,x,\xi)]R^{\varepsilon_k}\,(\lambda_{k-1},\lambda_k)\,\times$$
$$R\,\mathcal{A}[\Delta_{s^*,k_2}t,\, -\Delta_{s^*,k_2}x,\, \Delta_{s^*,k_2}\xi]\,(\lambda_{k_2},v).$$

We note that $R^{\varepsilon}\chi_0 R^{\varepsilon} = \chi_{\varepsilon}$, where the latter is from (2.7). Therefore,

$$F^{\vec{\varepsilon}}[k_1,k_2\,|\,(k_1,k_2]](r,u;s,v) = \mathbf{1}_{\{k_1<r^*,\, s^*<k_2,\, k_1<k_2\}}\, e^{\mu(v-u)}\, \mathcal{A}[\Delta_{k_1,r^*}t,\, -\Delta_{k_1,r^*}x,\, \Delta_{k_1,r^*}\xi]\,\chi_0 R\,\times$$
$$\prod_{k_1<k<k_2} \mathcal{A}[\Delta_k(t,x,\xi)]\,\chi_{\varepsilon_k}\, \mathcal{A}[\Delta_{k_2}(t,x,\xi)]\,\chi_0 R\,\mathcal{A}[\Delta_{s^*,k_2}t,\, -\Delta_{s^*,k_2}x,\, \Delta_{s^*,k_2}\xi]\,(u,v).$$

We now express all of the matrix kernels from Definition 2.1 like above. We will omit the conjugation factor $e^{\mu(v-u)}$ and the variables u, v from these expressions. Let us also use the shorthand $\Delta_{a,b}(t,-x,\xi) = (\Delta_{a,b}t,\, -\Delta_{a,b}x,\, \Delta_{a,b}\xi)$. We then have the following.

(1) $F[p\,|\,p](r,s) = \mathbf{1}_{\{r=p\}}\, R\mathcal{A}[\Delta_p(t,x,\xi)]\,\chi_0 R\,\mathcal{A}[\Delta_{s^*,p}(t,-x,\xi)]$.

(2) $F[k,k\,|\,\emptyset](r;s) = \mathbf{1}_{\{s<k<r^*\}}\,\mathcal{A}[\Delta_{k,r^*}(t,-x,\xi)]\,\chi_1\,\mathcal{A}[\Delta_{s,k}(t,-x,\xi)]$.

(3) $F[p,k\,|\,p](r,s) = \mathbf{1}_{\{r=p,\, s<k<p\}}\, R\mathcal{A}[\Delta_p(t,x,\xi)]\,\chi_0 R\,\mathcal{A}[\Delta_{k,p}(t,-x,\xi)]\,\chi_0\,\mathcal{A}[\Delta_{s,k}(t,-x,\xi)]$.

(4) $F[k_1,k_1,k_2\,|\,\emptyset](r;s) = \mathbf{1}_{\{k_1<r^*,\, s<k_2<k_1\}}\,\mathcal{A}[\Delta_{k_1,r^*}(t,-x,\xi)]\,\chi_1\,\mathcal{A}[\Delta_{k_2,k_1}(t,-x,\xi)]\,\chi_0\,\mathcal{A}[\Delta_{s,k_2}(t,-x,\xi)]$.

(5) $F^{\vec{\varepsilon}}[k_1\,|\,(k_1,k_2]](r;s) = \mathbf{1}_{\{k_1<r^*,\, s=k_2<p,\, k_1<k_2\}}\,\mathcal{A}[\Delta_{k_1,r^*}(t,-x,\xi)]\,\chi_0 R\,\times$
$$\prod_{k_1<k<k_2} \mathcal{A}[\Delta_k(t,x,\xi)]\,\chi_{\varepsilon_k}\,\mathcal{A}[\Delta_{k_2}(t,x,\xi)]\,R.$$

(7) $F^{\vec{\varepsilon}}[k_1,k_2,k_3\,|\,(k_1,k_2]](r;s) = \mathbf{1}_{\{k_1<r^*,\, s<k_3<k_2,\, k_1<k_2\}}\,\mathcal{A}[\Delta_{k_1,r^*}(t,-x,\xi)]\,\chi_0 R\,\times$
$$\prod_{k_1<k<k_2} \mathcal{A}[\Delta_k(t,x,\xi)]\,\chi_{\varepsilon_k}\,\mathcal{A}[\Delta_{k_2}(t,x,\xi)]\,\chi_0 R\,\mathcal{A}[\Delta_{k_3,k_2}(t,-x,\xi)]\,\chi_0\,\mathcal{A}[\Delta_{s,k_2}(t,-x,\xi)].$$

3. Discrete considerations: multi-point distribution function

In this section we derive a determinantal expression for the probability in (1.2). As $\mathbf{G}(m,n)$ depends only on the values of $\mathbf{G}$ to the left or below (m,n), the joint law of $\mathbf{G}(m_1,n_1),\dots,\mathbf{G}(m_p,n_p)$ depends on the restriction of $\mathbf{G}$ to $[0,m_p]\times[0,n_p]$.

Let us set $N=n_p$ throughout this section. Define the vector

$$\vec{\mathbf{G}}(m)=\big(\mathbf{G}(m,1),\mathbf{G}(m,2),\dots,\mathbf{G}(m,N)\big)\quad \text{for } \ m\geqslant 0.$$

The process $\vec{\mathbf{G}}(m)$ is a Markov chain by definition. It turns out to have an explicit transition rule.

3.1. Markov transition rule.

Let ∇ be the finite difference operator acting on $f:\mathbb{Z}\to\mathbb{C}$ as

$$\nabla f(x)=f(x+1)-f(x). \tag{3.1}$$

The operator has as inverse given by

$$\nabla^{-1}f(x)=\sum_{y<x}f(y), \tag{3.2}$$

valid so long as f vanishes identically to the left of some integer. This will be the case for functions that we consider. Since ∇f and $\nabla^{-1}f$ are then also functions of the same type, we may consider integer powers of ∇ acting on such functions.

Define the negative binomial weight

$$w_m(x)=\binom{x+m-1}{x}(1-q)^m q^x\,\mathbf{1}_{\{x\geqslant 0\}}\quad \text{for } \ m\geqslant 1 \ \text{ and } \ x\in\mathbb{Z}.$$

This is the probability of observing the m-th head at $x+m$ tosses of a coin that lands heads with probability $1-q$. It is a probability density, being the $(0,x)$-entry of $\big(I-\frac{q}{1-q}\nabla\big)^{-m}$.

Define also

$$\mathbb{W}_N=\{(x_1,\dots,x_N)\in\mathbb{Z}^N: x_1\leqslant\cdots\leqslant x_N\},$$

noting that $\vec{\mathbf{G}}$ takes values in $\mathbb{W}_N$.

Proposition 3.1. *The process $\vec{\mathbf{G}}(m)$ is a Markov chain with transition rule*

$$\mathbf{Pr}\left[\vec{\mathbf{G}}(m)=\mathbf{y}\mid\vec{\mathbf{G}}(\ell)=\mathbf{x}\right]=\det\Big(\nabla^{j-i}w_{m-\ell}(y_j-x_i)\Big)_{i,j} \tag{3.3}$$

for every $\mathbf{x},\mathbf{y}\in\mathbb{W}_N$ and $m>\ell$.

The proposition is proved in [21] following the paper [40] by Warren. It is related to determinantal expressions for non-intersecting path probabilities that appear in Karlin-McGregor or Lindström-Gessel-Viennot type arguments. The paths in this case are trajectories of the components of $\vec{\mathbf{G}}(m)$. The transition matrix of this chain turns out to be intertwined with a Karlin-McGregor type matrix by way of an RSK mechanism, which allows calculation of the former. The papers [15, 31] also give a systematic exposition to such computations.

Remark 3.1. Formula (3.3) has very similar structure to Schütz type formulas [3, 36, 37] for the transition rule of $\vec{\mathbf{G}}$. Schütz's formula for the N-particle continuous time TASEP $\mathbf{X}(t)$ is

$$\mathbf{Pr}\left[\mathbf{X}(t)=\mathbf{y}\,|\,\mathbf{X}(0)=x\right]=\det\left(\nabla^{j-i}F_t(\tilde{y}_j-\tilde{x}_i)\right)_{i,j}$$

where $F_t(x)=\frac{e^{-t}t^x}{x!}\mathbf{1}_{\{x\geqslant 0\}}$ is the Poisson density. Here the finite difference operator ∇ means $\nabla f(x)=f(x)-f(x+1)$, and its inverse is $\nabla^{-1}f(x)=\sum_{y\geqslant x}f(y)$. Particle locations are ordered such that $x_1>x_2>\cdots>x_N$, we let $\tilde{x}_j=x_{N+1-j}$, and likewise for $\mathbf{y}$.

A similar formula holds for the discrete time N-particle TASEP with sequential updates (see [15, 34]), where the rightmost particle attempts to jump first with probability q, followed by the particle to its left, and so on. The transition rule above is then modified by replacing $F_t(x)$ with the binomial density $F_{t,q}(x)=(1-q)^{-1}w_{t-x+1}(x)$. With parallel updates, discrete time TASEP becomes equivalent to the discrete polynuclear growth model as explained, for instance, in [4, 24].

Denote Pr the probability (1.2) that $\mathbf{G}(m_r,n_r)<a_r$ for every r. By proposition 3.1,

$$\mathrm{Pr}=\sum_{\substack{x^1,\dots,x^p\in\mathbb{W}_N\\ x^r_{n_r}<a_r}}\prod_{r=1}^{p}\det\left(\nabla^{j-i}w_{m_r-m_{r-1}}(x^r_j-x^{r-1}_i)\right)_{i,j} \tag{3.4}$$

with the convention that $x^0=0$. We will drop subscripts i,j from the determinants since all of them will be of $N\times N$ matrices with rows indexed by i and columns by j.

Lemma 3.1. *Recall the Δ_k notation: $\Delta_k y=y_k-y_{k-1}$ for $y=n,m$. The sum* (3.4) *simplifies to*

$$\mathrm{Pr}=\sum_{\substack{x^1,\dots,x^{p-1}\in\mathbb{W}_N\\ x^r_{n_r}<a_r}}\det\left(\nabla^{n_1-i}w_{m_1}(x^1_j)\right)\prod_{r=2}^{p-1}\det\left(\nabla^{\Delta_r n}w_{\Delta_r m}(x^r_j-x^{r-1}_i)\right)\times \tag{3.5}$$
$$\times\det\left(\nabla^{j-1-n_{p-1}}w_{\Delta_p m}(a_p-x^{p-1}_i)\right).$$

Proving this is the subject of the next section.

3.2. **Summation by parts.** The following is Lemma 3.2 in [22] and related to Lemma 3.2 in [21].

Lemma 3.2. *Let $f,g:\mathbb{Z}\to\mathbb{C}$ be such that $f(x)=g(x)=0$ if $x<L$ (typically L is very negative). Let $a_i,b_i\in\mathbb{Z}$ for $i=1,\dots,N$, and consider k such that $1\leqslant k\leqslant N$. Then,*

$$\sum_{\substack{x\in\mathbb{W}_N\\ x_k<A}}\det\left(\nabla^{j-a_i}f(x_j-y_i)\right)\det\left(\nabla^{b_j-i}g(z_j-x_i)\right) \tag{3.6}$$
$$=\sum_{\substack{x\in\mathbb{W}_N\\ x_k<A}}\det\left(\nabla^{k-a_i}f(x_j-y_i)\right)\det\left(\nabla^{b_j-k}g(z_j-x_i)\right).$$

Moreover,

$$\sum_{\substack{z\in\mathbb{W}_N\\ z_N<A}}\det\left(\nabla^{j-a_i}g(z_j-x_i)\right)=\det\left(\nabla^{j-1-a_i}g(A-x_i)\right). \tag{3.7}$$

It is instructive to understand the proof of this lemma, so we will outline the argument. It should be contrasted with the approach in [36], see also [3], which manipulates determinants by using that ∇^{-1} is a summation operator.

Proof. For identity (3.7), first note that $\sum_{x=a}^{b-1} \nabla f(x) = f(b) - f(a)$. Now perform the summation from z_N down to z_1, using multi-linearity of the determinant, which reduces ∇ by 1 in the corresponding column. After each step one finds a difference of two determinants, and the one with minus sign is zero due to two consecutive columns being equal. After the z_1-sum, the determinant with minus sign is zero because its first column stabilizes to zero as $z_1 \to -\infty$. For example, during the summation over z_N we have

$$\sum_{\substack{z \in \mathbb{W}_{N-1} \\ z_{N-1} < A}} \sum_{z_N = z_{N-1}}^{A-1} \det\left(\nabla^{1-a_i} g(z_1 - x_i) \cdots \nabla^{N-1-a_i} g(z_{N-1} - x_i)\, \nabla^{N-a_i} g(z_N - x_i)\right) =$$

$$\sum_{\substack{z \in \mathbb{W}_{N-1} \\ z_{N-1} < A}} \det\left(\nabla^{1-a_i} g(z_1 - x_i) \cdots \nabla^{N-1-a_i} g(z_{N-1} - x_i)\, \nabla^{N-1-a_i} g(A - x_i)\right) -$$

$$\det\left(\nabla^{1-a_i} g(z_1 - x_i) \cdots \nabla^{N-1-a_i} g(z_{N-1} - x_i)\, \nabla^{N-1-a_i} g(z_{N-1} - x_i)\right) =$$

$$\sum_{\substack{z \in \mathbb{W}_{N-1} \\ z_{N-1} < A}} \det\left(\nabla^{1-a_i} g(z_1 - x_i) \cdots \nabla^{N-1-a_i} g(z_{N-1} - x_i)\, \nabla^{N-1-a_i} g(A - x_i)\right).$$

Identity (3.6) in based on the following idea. First, it is enough to establish it for the sum over $\{x \in \mathbb{W}_N : x_k = A\}$. Suppose $[a_{i,j}]$ is a square matrix, the ℓ-th column of which has the from $a_{i,\ell} = \nabla f_{i,\ell}(x_\ell)$, and variable x_ℓ appears nowhere else. Then $\det\left(a_{i,j}\right) = \nabla_\ell \det\left(a_{i,1} \cdots f_{i,\ell}(x_\ell) \cdots\right)$, where ∇_ℓ is the difference operator in the x_ℓ variable. Now recall the summation by parts identity:

$$\sum_{x=a}^{b} u(x) \nabla[v(-x)] = \sum_{x=a}^{b} \nabla u(x) v(-x) + u(b+1) v(-b) - u(a) v(-a+1).$$

Combining these we have the following. Suppose $c_j, d_j \in \mathbb{Z}$ are such that for an index $\ell > k$, $c_\ell = c_{\ell+1}$ if $\ell < N$ and $d_{\ell-1} = d_\ell - 1$. Define $d_j^- = d_j - \mathbf{1}_{\{j=\ell\}}$ and $c_j^- = c_j - \mathbf{1}_{\{j=\ell\}}$. Then,

$$\begin{aligned} &\sum_{\substack{x \in \mathbb{W}_N \\ x_k = A}} \det\left(\nabla^{d_j - a_i} f(x_j - y_i)\right) \det\left(\nabla^{b_j - c_i} g(z_j - x_i)\right) \\ &= \sum_{\substack{x \in \mathbb{W}_N \\ x_k = A}} \det\left(\nabla^{d_j^- - a_i} f(x_j - y_i)\right) \det\left(\nabla^{b_j - c_i^-} g(z_j - x_i)\right). \end{aligned} \tag{3.8}$$

In plain words, one can move a derivative from column ℓ of the first determinant to the second's, decreasing d_ℓ and c_ℓ by 1 as a result. Indeed, consider the sum over variable x_ℓ on the l.h.s. of (3.8) while holding the other variables fixed. Upon transposing the second matrix and using the aforementioned observations in order, we see that

$$\sum_{x_\ell = x_{\ell-1}}^{x_{\ell+1}} \det\left(\nabla^{d_j - a_i} f(x_j - y_i)\right) \det\left(\nabla^{b_i - c_j} g(z_i - x_j)\right)$$
$$= \sum_{x_\ell = x_{\ell-1}}^{x_{\ell+1}} \det\left(\nabla^{d_j^- - a_i} f(x_j - y_i)\right) \det\left(\nabla^{b_j - c_i^-} g(z_j - x_i)\right) + (\text{boundary term}).$$

If $\ell = N$ then $x_{\ell+1} = +\infty$, and if $\ell = 1$ then $x_{\ell-1} = -\infty$. The boundary term equals $(\mathrm{I}) - (\mathrm{II})$, where

$$(\mathrm{I}) = \det\left(\nabla^{d_j^- - a_i} f(x_j - y_i)\right)\Big|_{x_\ell := x_{\ell+1}+1} \cdot \det\left(\nabla^{b_i - c_j} g(z_i - x_j)\right)\Big|_{x_\ell := x_{\ell+1}}$$
$$(\mathrm{II}) = \det\left(\nabla^{d_j^- - a_i} f(x_j - y_i)\right)\Big|_{x_\ell := x_{\ell-1}} \cdot \det\left(\nabla^{b_i - c_j} g(z_i - x_j)\right)\Big|_{x_\ell := x_{\ell-1}-1}.$$

The term $(\mathrm{I}) = 0$ because the ℓ and $(\ell+1)$-st column of the second determinant agree due to $c_\ell = c_{\ell+1}$ when $\ell < N$. If $\ell = N$ then it is 0 because $\nabla^m g(z - x) = 0$ for all sufficiently large x, which makes the last column of the second determinant 0. The term $(\mathrm{II}) = 0$ for the same reason with respect to the first determinant since $d_{\ell-1} = d_\ell - 1$.

Analogously, for an $\ell < k$, suppose $c_{\ell+1} = c_\ell + 1$ and $d_\ell = d_{\ell-1}$ if $\ell > 1$. Then we may move a derivative from the ℓ-th column of the first determinant to the second's in the l.h.s. of (3.8), which will result in c_ℓ and d_ℓ being increased by 1.

Identity (3.6) follows by first applying (3.8) to columns $\ell = N, N-1, \dots, k+1$ *in that order*. The conditions on c_ℓ and d_ℓ are then satisfied during each application. Then we apply (3.8) to $\ell = N, \dots, k+2$, followed by to $\ell = N, \dots, k+3$, and so on. The derivative in column $j > k$ is reduced by $j - k$. Similarly, we apply the derivative incrementing procedure first to columns $\ell = 1, \dots, k-1$, then to columns $\ell = 1, \dots, k-2$, and so forth to increase the derivative in column $j < k$ by $k - j$. ∎

Proof of Lemma 3.1. In order to simplify the expression for Pr from (3.4) we apply Lemma 3.2 iteratively. Apply (3.6) to the expression (3.4) with respect to the sum over x^1, which involves the first two determinants. In doing so, set $k = n_1$, $a_i = i$, $b_j = j$, $f = w_{n_1}$, etc. We find that

$$\mathrm{Pr} = \sum_{\substack{x^1, \dots, x^p \in \mathbb{W}_N \\ x^r_{n_r} < a_r}} \det\left(\nabla^{n_1 - i} w_{m_1}(x^1_j)\right) \det\left(\nabla^{j - n_1} w_{\Delta_2 m}(x^2_j - x^1_i)\right) \times$$
$$\times \prod_{r=3}^{p} \det\left(\nabla^{j-i} w_{\Delta_r m}(x^r_j - x^{r-1}_i)\right).$$

Next, apply (3.6) to the sum over x^2 – involving the 2nd and 3rd determinants – with $k = n_2$ and $a_i \equiv n_1$. Then,

$$\Pr = \sum_{\substack{x^1,\dots,x^p \in \mathbb{W}_N \\ x^r_{n_r} < a_r}} \det\left(\nabla^{n_1 - i} w_{m_1}(x^1_j)\right) \det\left(\nabla^{\Delta_2 n} w_{\Delta_2 m}(x^2_j - x^1_i)\right) \det\left(\nabla^{j-n_2} w_{\Delta_3 m}(x^3_j - x^2_i)\right)$$
$$\times \prod_{r=4}^{p} \det\left(\nabla^{j-i} w_{\Delta_r m}(x^r_j - x^{r-1}_i)\right).$$

After iterating like this for all the variables, we finally use (3.7) to perform the sum over x^p with $x^p_N < a_p$ (recall $n_p = N$). This gives the expression (3.5). ■

We would like to express $\Pr$ as a single $N \times N$ determinant. This would ordinarily be done by using the Cauchy-Binet identity iteratively over each of the sums. However, the constraints $x^r_{n_r} < a_r$ prevent a direct application. This is addressed in the following section.

3.3. **Cauchy-Binet identity.** Let us manipulate the expression from (3.5) in the following way. First, consider $N \times N$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ such that $\det(A) \cdot \det(B) = 1$. In fact, we will chose A and B to be triangular with $a_{ii} = b^{-1}_{ii}$. We multiply the matrix of the first determinant from (3.5) by A and of the last one by B. Doing so will set us up for the orthogonalization procedure of the next section.

Formally, introduce functions $f_{0,1}, f_{1,2}, \dots, f_{p-1,p}$ as follows. We assume that $p \geqslant 2$. When $p = 1$ we can use a simpler approach as explained in §4.3. For $1 \leqslant i, j \leqslant N$ as well as $x, y \in \mathbb{Z}$,

$$f_{0,1}(i, x) = \sum_{k=1}^{N} a_{ik}\, \nabla^{n_1 - k} w_{m_1}(x + a_1) \cdot (-1)^{n_1}, \tag{3.9}$$
$$f_{r-1,r}(x, y) = \nabla^{\Delta_r n} w_{\Delta_r m}(y - x + \Delta_r a) \cdot (-1)^{\Delta_r n} \quad \text{for } 1 < r < p,$$
$$f_{p-1,p}(x, j) = \sum_{k=1}^{N} \nabla^{k-1-n_{p-1}} w_{\Delta_p m}(\Delta_p a - x)\, b_{kj} \cdot (-1)^{n_{p-1}}.$$

Then $\Pr$ equals

$$\Pr = \sum_{\substack{x^1,\dots,x^{p-1} \in \mathbb{W}_N \\ x^k_{n_k} < 0}} \det\left(f_{0,1}(i, x^1_j)\right) \prod_{k=2}^{p-1} \det\left(f_{k-1,k}(x^{k-1}_i, x^k_j)\right) \det\left(f_{p-1,p}(x^{p-1}_i, j)\right). \tag{3.10}$$

The summation constraints became $x^k_{n_k} < 0$ because we have shifted $x^k_i \mapsto x^k_i + a_r$ in defining $f_{k-1,k}$. Also, the powers of -1 in the fs do not change the product of the determinants because they factor out as $(-1)^{N\cdot(n_1 + \Delta_2 n + \cdots + \Delta_{p-1} n + n_{p-1})} = (-1)^{2N n_{p-1}}$.

Consider $\theta = (\theta_1, \dots, \theta_{p-1})$ where each $\theta_k \in \mathbb{C} \setminus 0$. Define an $N \times N$ matrix $L = L(i, j \,|\, \theta)$ as follows with the convention that $\theta^0_k = 1$.

$$L(i, j \,|\, \theta) = \sum_{(x_1,\dots,x_{p-1}) \in \mathbb{Z}^{p-1}} f_{0,1}(i, x_1) \prod_{k=2}^{p-1} f_{k-1,k}(x_{k-1}, x_k)\, f_{p-1,p}(x_{p-1}, j) \prod_{k=1}^{p-1} \theta_k^{\mathbf{1}_{\{x_k < 0\}} - \mathbf{1}_{\{i \leqslant n_k\}}}. \tag{3.11}$$

The sum is actually finite because $f_{r-1,r}(x,y)$ vanishes for all sufficiently large x or small y. Apart from the factors involving θ, L is the convolution $f_{0,1} * \cdots * f_{p-1,p}$ or, if we think of the fs as matrix kernels, then it is the product $f_{0,1} \cdots f_{p-1,p}$. The conclusion of this section is

Lemma 3.3. *Let γ_r be a counterclockwise circular contour of radius $r > 1$. Set $\gamma_r^{p-1} = \overbrace{\gamma_r \times \cdots \times \gamma_r}^{p-1}$.*

$$\Pr = \oint\limits_{\gamma_r^{p-1}} d\theta_1 \cdots d\theta_{p-1} \, \frac{\det\left(L(i,j \,|\, \theta)\right)}{\prod_{k=1}^{p-1}(\theta_k - 1)}. \tag{3.12}$$

Proof. For $x \in \mathbb{W}_N$, the condition $x_n < 0$ is equivalent to $\#\{x_j < 0\} \geqslant n$. Now for $\ell \in \mathbb{Z}$,

$$\mathbf{1}_{\{\ell \geqslant 0\}} = \oint\limits_{\gamma_r} d\theta \, \frac{\theta^{\ell}}{\theta - 1}.$$

Consequently,

$$\mathbf{1}_{\{\#\{x_j<0\} \geqslant n\}} = \oint\limits_{\gamma_r} d\theta \, \frac{\prod_{j=1}^{N} \theta^{\mathbf{1}_{\{x_j<0\}}}}{\theta^n(\theta - 1)}. \tag{3.13}$$

If we apply (3.13) to the expression (3.10) for $\Pr$ we find

$$\Pr = \oint\limits_{\gamma_r^{p-1}} d\theta_1 \cdots d\theta_{p-1} \prod_{k=1}^{p-1} \frac{\theta_k^{-n_k}}{\theta_k - 1} \Bigg[\sum_{\substack{x^k \in \mathbb{W}_n \\ 1 \leqslant k < p}} \det\left(f_{0,1}(i, x_j^1)\, \theta_1^{\mathbf{1}_{\{x_j^1<0\}}} \right) \times$$
$$\prod_{k=2}^{p-1} \det\left(f_{k-1,k}(x_i^{k-1}, x_j^k)\, \theta_k^{\mathbf{1}_{\{x_j^k<0\}}} \right) \det\left(f_{p-1,p}(x_i^{p-1}, j) \right) \Bigg].$$

We push $\theta_k^{-n_k}$ into the first determinant by inserting θ_k^{-1} into its first n_k rows. Then, by the Cauchy-Binet identity, the quantity inside square brackets is $\det\left(L(i,j \,|\, \theta)\right)$. ∎

Expression (3.12) is a discrete determinantal formula for the multi-point distributions functions (1.2). However, matrix L does not have good asymptotical behaviour for the KPZ scaling limit (or numerical estimates). It is necessary to express $\det(L)$ as a Fredholm determinant over a space free of parameter N. This is the subject of the following section.

4. Orthogonalization: representation as a Fredholm determinant

Recall the triangular matrices A and B from §3.3. Multiplication by them is essentially performing elementary row and column operations, which is an orthogonalization procedure. The entries of A and B, vaguely put, will be like inverses to entries of the first and last determinant in (3.5). These are obtained by extending $\nabla^n w_m(x)$ to negative m, which motivates the following. Later in §4.3 we provide intuition for this orthogonalization by explaining it for the single point law.

4.1. **Contour integrals.** Recall the functions G^* and G from (2.8) and (2.9). The 3-parameter family $G^*(\cdot\,|\,n,m,a)$ and $G(\cdot\,|\,n,m,a)$ form a group in that for $w \neq 0, 1-q, 1$:

$$\begin{aligned} G^*(w\,|\,n+n', m+m', a+a') &= G^*(w\,|\,n,m,a)\cdot G^*(w\,|\,n',m',a'), \\ G^*(w\,|\,-n,-m,-a) &= G^*(w\,|\,n,n,a)^{-1}, \\ G^*(w\,|\,0,0,0) &= 1, \end{aligned} \tag{4.1}$$

and analogously for G. The *group property* will make it convenient to follow upcoming calculations and give further intuition for the orthogonalization procedure.

From the generating function $(1+z)^{-k} = \sum_{x\geqslant 0}\binom{-k}{x}z^x$ for negative binomials, it follows that

$$w_m(x) = \oint_{\gamma_\rho} dz\left(\frac{1-qz}{1-q}\right)^{-m} z^{-x-1},$$

where $\rho < 1$. Changing variables $z \mapsto (1-z)^{-1}$ gives a contour integral representation of $w_m(x)$ that, upon applying integer powers of ∇ according to (3.1) and (3.2), shows that

$$\nabla^n w_m(x) = (-1)^{n-1}\oint_{\gamma_r(1)} dz\, G^*(z\,|\,n,m,x-1) \tag{4.2}$$

with radius $r > 1$ (so $\gamma_r(1)$ encloses all possible poles at $z = 0, 1-q, 1$). The condition $r > 1$ ensures that the summation needed to apply ∇^{-1} to $G^*(z\,|\,n,m,x)$ in the x-variable, is legal throughout $z \in \gamma_r(1)$. The right hand side of (4.2) continues $\nabla^n w_m(x)$ to integer values of all parameters.

Define the matrices A and B as follows. Let $c(k)$ be the conjugation factor defined in (2.11), and recall $m(k)$ and $a(k)$ from (2.4). Consider any radius $\tau < 1-q$.

$$\begin{aligned} a_{ik} &= c(i)(-1)^k \oint_{\gamma_\tau} d\zeta\, \frac{1}{G^*\big(\zeta\,|\,i-k+1, m(i), a(i)-1\big)}, \\ b_{kj} &= c(j)^{-1}(-1)^k \oint_{\gamma_\tau} d\zeta\, \frac{1}{G^*\big(\zeta\,|\,k-j+1, m_p-m(j), a_p-a(j)\big)}. \end{aligned} \tag{4.3}$$

The matrices A and B are lower-triangular with $a_{ii} = c(i)(-1)^i = b_{ii}^{-1}$; so $\det(A)\det(B) = 1$. This is because

$$\oint_{\gamma_\tau} d\zeta\, \frac{1}{G^*(\zeta\,|\,n+1,m,a)} = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n < 0. \end{cases}$$

Lemma 4.1. *The following identities hold.*

(1) *If* $1 \leqslant i \leqslant N$ *and* $|z| > \tau$,

$$\oint_{\gamma_\tau} d\zeta\, \frac{1}{G^*(\zeta\,|\,i,m,a)\,(z-\zeta)} = \sum_{k=1}^{N}\oint_{\gamma_\tau} d\zeta\, \frac{z^{-k}}{G^*(\zeta\,|\,i-k+1,m,a)}.$$

(2) *If* $1 \leqslant j \leqslant N$ *and* $|z| > \tau$,

$$\oint_{\gamma_\tau} d\zeta \, \frac{z^{N+1}}{G^*(\zeta \,|\, N+1-j, m, a)\,(z-\zeta)} = \sum_{k=1}^{N} \oint_{\gamma_\tau} d\zeta \, \frac{z^k}{G^*(\zeta \,|\, k-j+1, m, a)}.$$

Proof. The first identity follows by expanding $(z-\zeta)^{-1}$ in powers of ζ/z. The contribution of terms on the r.h.s. with $k > i$ is zero. The second one follows from the first by re-indexing $k \mapsto N+1-k$ and substituting $i = N+1-j$. ∎

For the rest of this section we will deduce an expression for $L(i,j\,|\,\theta)$ in terms of contour integrals. Recalling the $f_{r-1,r}$s from (3.9), then (4.2) and (4.3), we infer the following.

$$f_{0,1}(i,x_1) = -c(i) \oint_{\gamma_{\tau_1}} d\zeta_1 \oint_{\gamma_{R_1}(1)} dz_1 \, \frac{G^*\big(z_1 \,|\, n_1, m_1, a_1+x_1-1\big)}{G^*\big(\zeta_1 \,|\, i, m(i), a(i)-1\big)\,(z_1-\zeta_1)}\,,$$

$$f_{r-1,r}(x_{r-1},x_r) = -\oint_{\gamma_{R_r}(1)} dz_r \, G^*(z_r \,|\, \Delta_r n, \Delta_r m, \Delta_r a - 1) \quad \text{for } 1 < r < p,$$

$$f_{p-1,p}(x_{p-1},j) = c(j)^{-1} \oint_{\gamma_{\tau_2}} d\zeta_2 \oint_{\gamma_{R_p}(1)} dz_p \, \frac{G^*\big(z_p \,|\, \Delta_p n, \Delta_p m, \Delta_p a - x_{p-1}-1\big)}{G^*\big(\zeta_2 \,|\, n_p-j+1, m_p-m(j), a_p-a(j)\big)\,(z_p-\zeta_2)}\,.$$

The contours above are circular and arranged as follows. Contours γ_{τ_1} and γ_{τ_2} are around the origin with $\tau_2 < \tau_1 < 1-q$ (τ_1 and τ_2 are ordered for definiteness). Contours $\gamma_{R_k}(1)$ are around 1 with every $R_k > 1+\tau_1$, that is, they enclose the contours around the origin and the numbers $0, 1-q, 1$. In deriving expressions for $f_{0,1}$ and $f_{p-1,p}$ we have used Lemma 4.1.

Upon multiplying all the fs we get $(-1)^{p-1}c(i)c(j)^{-1} \times \big(\text{a } (p+2)-\text{fold contour integral}\big)$. In this integral we would like to replace every G^* by the corresponding G. In doing so we obtain factors of $G^*(1-\sqrt{q}\,|\,\cdot,\cdot,\cdot)$, which, by the group property of G^*, multiply to

$$G^*\big(1-\sqrt{q}\,|\, j-i-1, m(j)-m(i), a(j)-a(i)\big).$$

When multiplied by $c(i)c(j)^{-1}$ this equals $c(i,j)/(1-\sqrt{q})$, where $c(i,j)$ is the conjugation factor (2.12).

We may plug the product above into the definition of $L(i,j\,|\,\theta)$ from (3.11). There we have a sum over $\vec{x} \in \mathbb{Z}^{p-1}$ and a product involving θ. Let us write the product of θ_ks as follows, recalling $\chi_1(x) = \mathbf{1}_{\{x<0\}}$ and $\chi_2(x) = \mathbf{1}_{\{x \geqslant 0\}}$ from (2.7). Note $\theta^{\mathbf{1}_{\{x<0\}}} = \theta^{2-1}\chi_1(x) + \theta^{2-2}\chi_2(x)$. Therefore,

$$\begin{aligned}\prod_{k=1}^{p-1} \theta_k^{\mathbf{1}_{\{x_k<0\}}-\mathbf{1}_{\{i\leqslant n_k\}}} &= \sum_{\vec{\varepsilon}\in\{1,2\}^{p-1}} \prod_{k=1}^{p-1} \theta_1^{2-\varepsilon_k-\mathbf{1}_{\{i\leqslant n_k\}}} \cdot \chi_{\varepsilon_1}(x_1)\cdots\chi_{\varepsilon_{p-1}}(x_{p-1}) \\ &= \sum_{\vec{\varepsilon}\in\{1,2\}^{p-1}} \theta^{\vec{\varepsilon}}(i)\,\chi_{\vec{\varepsilon}}(\vec{x}),\end{aligned}$$

where $\chi_{\vec{\varepsilon}}(\vec{x}) = \prod_{k=1}^{p-1} \chi_{\varepsilon_k}(x_k)$ and $\theta^{\vec{\varepsilon}}(i)$ is notation from (2.5). From this expression we find that

$$L(i,j\,|\,\theta) = \sum_{\vec{\varepsilon}\in\{1,2\}^{p-1}} \frac{(-1)^{p-1}c(i,j)}{1-\sqrt{q}}\, \theta^{\vec{\varepsilon}}(i)\, L^{\vec{\varepsilon}}(i,j), \tag{4.4}$$

where $L^{\vec{\varepsilon}}(i,j)$ is the sum over $x \in \mathbb{Z}^{p-1}$ of $\chi_{\vec{\varepsilon}}(x)$ times the aforementioned $(p+2)$-fold contour integral.

Lemma 4.2. *Given $\vec{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_{p-1}) \in \{1,2\}^{p-1}$, $L^{\vec{\varepsilon}}(i,j)$ has the following contour integral form. Consider radii $\tau_2 < \tau_1 < 1-q$, as well as radii $R_1, \ldots, R_p$ such that every $R_k > 1+\tau_1$ and they satisfy the following pairwise ordering.*

$$R_k < R_{k+1} \ \text{if} \ \varepsilon_k = 2 \quad \text{while} \quad R_k > R_{k+1} \ \text{if} \ \varepsilon_k = 1. \tag{4.5}$$

There is such a choice of radii, and given these,

$$L^{\vec{\varepsilon}}(i,j) = (-1)^{\varepsilon_1+\cdots+\varepsilon_{p-1}} \oint_{\gamma_{\tau_1}} d\zeta_1 \oint_{\gamma_{\tau_2}} d\zeta_2 \oint_{\gamma_{R_1}(1)} dz_1 \cdots \oint_{\gamma_{R_p}(1)} dz_p$$
$$\frac{\prod_{k=1}^{p} G\big(z_k\,|\,\Delta_k(n,m,a)\big) \prod_{k=1}^{p-1} \left(z_k - z_{k+1}\right)^{-1} \left(\frac{1-\zeta_1}{1-z_1}\right)}{G\big(\zeta_1\,|\,i, m(i), a(i)\big)\, G\big(\zeta_2\,|\,n_p - j + 1, m_p - m(j), a_p - a(j)\big)\,(z_1-\zeta_1)\,(z_p-\zeta_2)}.$$

Proof. From the discussion preceeding the lemma we see that

$$L^{\vec{\varepsilon}}(i,j) = \sum_{(x_1,\ldots,x_{p-1})\in\mathbb{Z}^{p-1}} \oint_{\gamma_{\tau_1}} d\zeta_1 \oint_{\gamma_{\tau_2}} d\zeta_2 \oint_{\gamma_{R_1}(1)} dz_1 \cdots \oint_{\gamma_{R_p}(1)} dz_p\ \chi_{\varepsilon_1}(x_1)\cdots\chi_{\varepsilon_{p-1}}(x_{p-1})$$
$$\frac{\prod_{k=1}^{p-1} G(z_k\,|\,\Delta_k n, \Delta_k m, \Delta_k a + \Delta_k x - 1)\, G(z_p\,|\,\Delta_p n, \Delta_p m, \Delta_k a - x_{p-1} - 1)}{G\big(\zeta_1\,|\,i, m(i), a(i)-1\big)\, G\big(\zeta_2\,|\,n_p - j+1,\ m_p - m(j),\ a_p - a(j)\big)\,(z_1-\zeta_1)\,(z_p-\zeta_2)}.$$

From the group property, $G(z\,|\,n,m,a+x-1) = G(z\,|\,n,m,a)(1-z)^{x-1}$. Using this, we factor out every $(1-z_k)^{\Delta_k x - 1}$, $(1-z_p)^{-x_{p-1}-1}$ and $(1-\zeta_1)^{-1}$. Their contribution is

$$\prod_{k=1}^{p-1} \left(\frac{1-z_k}{1-z_{k+1}}\right)^{x_k} \cdot \frac{1-\zeta_1}{\prod_{k=1}^{p}(1-z_k)}.$$

Now suppose $z \in \gamma_{\rho_1}(1)$, $w \in \gamma_{\rho_2}(1)$ and $\varepsilon \in \{1,2\}$. Then,

$$\sum_{x\in\mathbb{Z}} \chi_\varepsilon(x) \left(\frac{1-z}{1-w}\right)^x = (-1)^\varepsilon\, \frac{1-w}{z-w},$$

so long as $\rho_1 < \rho_2$ in the case $\varepsilon = 2$ or $\rho_1 > \rho_2$ in the case $\varepsilon = 1$. The radii $R_1, \ldots, R_p$ have been chosen precisely to satisfy these constraints imposed by $\vec{\varepsilon}$. That it is possible to do so may be seen by induction on p as follows.

The base case of $p = 2$ is trivial. Now suppose there is an arrangement of radii $R_1, \ldots, R_p$ that satisfy the constraints given by $\varepsilon_1, \ldots, \varepsilon_{p-1}$, and we introduce an $\varepsilon_p \in \{1,2\}$. Find previous radii R_a and R_b such that $R_a < R_p < R_b$ (one of these may be vacuous). Now choose any radius $R_{p+1} > 1+\tau_1$ such that if $\varepsilon_p = 1$ then $R_a < R_{p+1} < R_p$, while if $\varepsilon_p = 2$ then have $R_p < R_{p+1} < R_b$.

This proves the claim. An explicit choice of such radii is the following:

$$R_1 \text{ satisfies } R_1 \cdot \big(1 - \frac{1}{2} - \cdots - \frac{1}{2^{p-1}}\big) > 1 + \tau_1; \quad R_k = R_1 \cdot \big(1 + \sum_{j=1}^{k-1} \frac{(-1)^{\varepsilon_j}}{2^j}\big). \tag{4.6}$$

Finally, using the summation identity above to carry out the sum over every x_k, and simplifying the resulting integrand, we get the representation of $L^{\vec{\varepsilon}}(i,j)$ stated in the lemma. ■

We conclude the section with a presentation of $L(i,j\,|\,\theta)$ that will be used to get a Fredholm determinant form in the next section and also for its asymptotics. Consider the contour integral form of $L^{\vec{\varepsilon}}(i,j)$ in Lemma 4.2. Deform each contour $\gamma_{R_k}(1)$ to a union of a contour around 0, say $\gamma_{\rho_k}(0)$, and a contour around 1, say $\gamma_{\rho'_k}(1)$. The first of these should enclose γ_{τ_1} and γ_{τ_2} and lie within the circle of radius $1-\sqrt{q}$. That is,

$$\tau_2 < \tau_1 < \rho_k < 1 - \sqrt{q} \;\text{ for every }\; k.$$

The second should enclose non-zero poles in variable z_k and lie outside the circle of radius $1-\sqrt{q}$. That is,

$$1 - \sqrt{q} < 1 - \rho'_k < 1 - q \;\text{ for every }\; k.$$

See Figure 1 for an illustration.

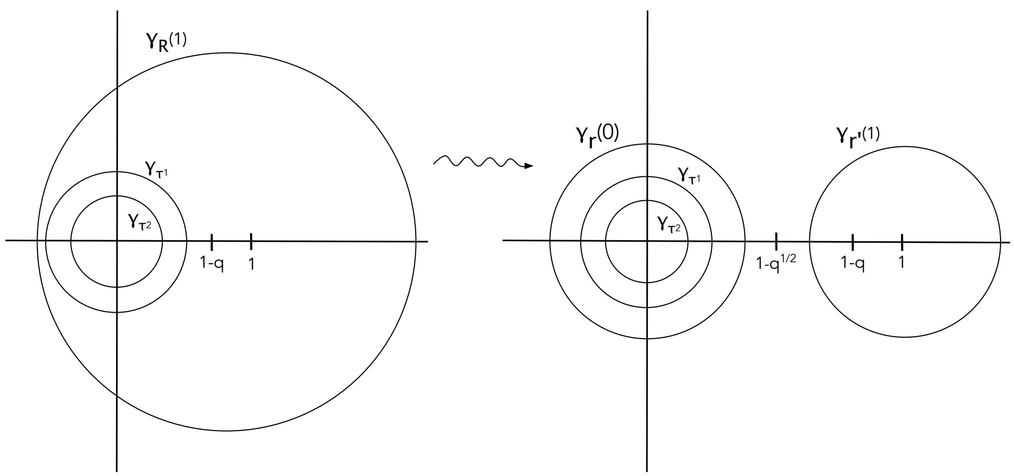

FIGURE 1. The deformation of $\gamma_R(1)$ into two contours $\gamma_r(0)$ and $\gamma_{r'}(1)$.

The radii of the contours should be arranged so that the ordering imposed by $\vec{\varepsilon}$ remains, that is, if $\varepsilon_k = 2$ then $\rho_k < \rho_{k+1}$ and $\rho'_k < \rho'_{k+1}$, etc. In order to simplify notation, we denote $\gamma_{\rho_k}(0)$ as $\gamma_{R_k}(0)$ and $\gamma_{\rho'_k}(1)$ as $\gamma_{R_k}(1)$. In this notation we write the contour integral for $L^{\vec{\varepsilon}}(i,j)$ as a sum of 2^p contour integrals, where for each integral we make a choice of contours $z_1 \in \gamma_{R_1}(\delta_1), z_2 \in \gamma_{R_2}(\delta_2), \ldots, z_p \in \gamma_{R_p}(\delta_p)$, and $\vec{\delta} = (\delta_1, \ldots, \delta_p) \in \{0,1\}^p$. Thus,

$$L(i,j\,|\,\theta) = \sum_{\vec{\delta}\in\{0,1\}^p} \sum_{\vec{\varepsilon}\in\{1,2\}^{p-1}} (-1)^{p-1+\varepsilon_1+\cdots+\varepsilon_{p-1}} \frac{c(i,j)}{1-\sqrt{q}}\, \theta^{\vec{\varepsilon}}(i)\, L^{\vec{\varepsilon}}_{\vec{\delta}}(i,j). \tag{4.7}$$

The entry $L^{\vec\varepsilon}_{\vec\delta}(i,j)$ looks the same as the integral in Lemma 4.2 except $\gamma_{R_k}(1)$ is replaced by $\gamma_{R_k}(\delta_k)$ in our simplified notation.

4.2. **Fredholm determinant form.** Looking at (4.7), the identity matrix in the Fredholm determinantal form for L will come from the contribution at $\vec\delta = \vec 0$. So we define some matrices by which the $L^{\vec\varepsilon}_{\vec\delta}$s will be expressed. Recall notations from §2.1.

Definition 4.1. Let $L_0 = 0$. For $1 \leqslant k \leqslant p$, define a matrix L_k as follows. For $1 \leqslant i,j \leqslant N$ (recall $N = n_p$),

$$L_k(i,j) = \frac{1}{1-\sqrt{q}} \oint\limits_{\gamma_{\tau_1}} d\zeta_1 \oint\limits_{\gamma_{\tau_2}} d\zeta_2 \, \frac{G\big(\zeta_1 \,|\, n_k - i, m_k - m(i), a_k - a(i)\big)}{G\big(\zeta_2 \,|\, n_k - j + 1, m_k - m(j), a_k - a(j)\big)\,(\zeta_1 - \zeta_2)}.$$

The radii should satisfy $\tau_2 < \tau_1 < 1 - \sqrt{q}$.

Definition 4.2. Suppose $0 \leqslant k_1 < k_2 \leqslant p$ and $\vec\varepsilon \in \{1,2\}^{p-1}$. Let $\tau_2 < \tau_1 < 1-\sqrt{q}$. Consider radii $R_{k_1+1}, \dots, R_{k_2}$ such that $q < R_k < \sqrt{q}$ for every k, and they are ordered in the following way:

$$R_k < R_{k+1} \ \text{ if } \ \varepsilon_k = 2 \quad \text{while} \quad R_k > R_{k+1} \ \text{ if } \ \varepsilon_k = 1.$$

Note this depends only on $\varepsilon_{k_1+1}, \dots, \varepsilon_{k_2-1}$. (It is possible to arrange the radii according to $\vec\varepsilon$ as shown in Lemma 4.2.) Set $\vec\gamma_{R^{\vec\varepsilon}} = \gamma_{R_{k_1+1}}(1) \times \cdots \times \gamma_{R_{k_2}}(1)$. Define a matrix $L^{\vec\varepsilon}_{(k_1,k_2]}$ as follows.

$$L^{\vec\varepsilon}_{(k_1,k_2]}(i,j) = \mathbf{1}_{\{i > n_{k_1}, j \leqslant n_{k_2}\}} \frac{1}{1-\sqrt{q}} \oint\limits_{\gamma_{\tau_1}} d\zeta_1 \oint\limits_{\gamma_{\tau_2}} d\zeta_2 \oint\limits_{\vec\gamma_{R^{\vec\varepsilon}}} dz_{k_1+1}\, dz_{k_1+2} \cdots dz_{k_2}$$

$$\frac{\prod_{k_1 < k \leqslant k_2} G\big(z_k \,|\, \Delta_k(n,m,a)\big) \prod_{k_1<k<k_2} (z_k - z_{k+1})^{-1} \left(\frac{1-\zeta_1}{1-z_1}\right)^{\mathbf{1}_{\{k_1=0\}}} (z_{k_1+1} - \zeta_1)^{-1} (z_{k_2} - \zeta_2)^{-1}}{G\big(\zeta_1 \,|\, i - n_{k_1}, m(i) - m_{k_1}, a(i) - a_{k_1}\big)\, G\big(\zeta_2 \,|\, n_{k_2} - j + 1, m_{k_2} - m(j), a_{k_2} - a(j)\big)}.$$

Lemma 4.3. *Suppose $\vec\delta$ is identically zero. Then $L^{\vec\varepsilon}_{\vec 0} = 0$ unless $\vec\varepsilon = (\overbrace{2,\dots,2}^{k-1}, \overbrace{1,\dots,1}^{p-k})$ for some k. In other words, it is the zero matrix unless there is a $k \in [1,p]$ such that the radii of contours $\gamma_{R_1}(0), \dots, \gamma_{R_p}(0)$ satisfy $R_1 < R_2 < \cdots < R_k$ and $R_p < \cdots < R_{k+1} < R_k$.*

Proof. The contour integral for $L^{\vec\varepsilon}_{\vec 0}$ has every contour arranged around the origin. The poles of the integrand in z-variables come from the term $(z_1 - \zeta_1)(z_p - \zeta_2) \prod_k (z_k - z_{k+1})$ in the denominator. Given $\vec\varepsilon$, suppose there is an index ℓ with $1 < \ell < p$ such that $R_\ell < R_{\ell-1}$ and $R_\ell < R_{\ell+1}$. Then we may contract the z_ℓ-contour without passing any poles in that variable. Hence, $L^{\vec\varepsilon}_{\vec 0}(i,j) = 0$. It follows that $L^{\vec\varepsilon}_{\vec 0}$ can only be non-zero if there is no such ℓ, which is the condition on ε in the lemma. ∎

Lemma 4.4. *Suppose $\vec\delta$ is not identically zero. Then $L^{\vec\varepsilon}_{\vec\delta} = 0$ unless $\vec\delta = (0,\dots,0,1,\dots,1,0,\dots,0)$, i.e., $\vec\delta$ consists of a run of 0s (possibly empty), followed by a run of 1s (non-empty), and ending with a run of 0s (again, possibly empty). Moreover, suppose $\vec\delta$ equals 1 for indices on the interval $(k_1,k_2]$ with $0 \leqslant k_1 < k_2 \leqslant p$. Then for $L^{\vec\varepsilon}_{\vec\delta}$ to be non-zero it must be that $\varepsilon_1 = \cdots = \varepsilon_{k_1-1} = 2$ and $\varepsilon_{k_2+1} = \cdots \varepsilon_{p-1} = 1$, i.e., $R_1 < \cdots < R_{k_1}$ and $R_{k_2+1} > \cdots > R_p$ (some of these may be vacuous).*

Proof. Given $\vec{\delta} = (\delta_1, \ldots, \delta_p)$ suppose there are indices $k_1 < k_2$ such that $\delta_{k_1} = 1$, $\delta_{k_1+1} = 0$ and $\delta_{k_2} = 1$. Consider the integral of $L^{\varepsilon}_{\delta}(i,j)$ involving the z_{k_1+1}-contour, which is around 0. As the z_{k_1}-contour is around 1, we may contract the z_{k_1+1}-contour to 0 unless the z_{k_1+2}-contour lies below it (around 0). But then we may contour that one unless the z_{k_1+3}-contour lies below it, and so on, until we get to the z_{k_2-1}-contour. In that case, we can always contact the z_{k_2-1}-contour because the z_{k_2}-contour is around 1. So $L^{\vec{\varepsilon}}_{\vec{\delta}}(i,j) = 0$ for such $\vec{\delta}$, which implies the condition on $\vec{\delta}$ in the lemma.

Now suppose $\vec{\delta} = (0, \ldots, 0, \overbrace{1, \ldots, 1}^{k_2-k_1}, 0, \ldots, 0)$. Consider the contours in the integral for $L^{\vec{\varepsilon}}_{\vec{\delta}}(i,j)$ in variables $z_1, \ldots, z_{k_1}$. They lie around 0 and we may contract the z_{k_1}-contour unless the z_{k_1-1}-contour lies below it, and so forth, which shows $L^{\vec{\varepsilon}}_{\vec{\delta}}(i,j) = 0$ unless $R_1 < R_2 < \cdots < R_{k_1}$. Similarly, it will be zero unless $R_p < \cdots < R_{k_2+1}$. This proves the condition stipulated on ε. ∎

Lemma 4.5. *For* $1 \leqslant k \leqslant p$, *set* $\varepsilon^k = (\overbrace{2, \ldots, 2}^{k-1}, \overbrace{1, \ldots, 1}^{p-k})$. *Then,* $L^{\varepsilon^k}_{\vec{0}} = (-1)^{k-1}(1-\sqrt{q})\big(L_k - L_{k-1}\big)$.

Proof. Look at the contour integral presentation of $L^{\vec{\varepsilon}}_{\vec{0}}(i,j)$ from Lemma 4.2. Since $\vec{\delta} = \vec{0}$, all contours are around the origin. We will contract the z-contours $\gamma_{R_1}, \ldots, \gamma_{R_p}$ in the order specified by ε^k, and use the group property of G to simplify the integrand. We have $R_1 < \cdots < R_k$ and $R_p < \cdots < R_{k+1} < R_k$.

First we contract the z_p-contour and pick up residue at $z_p = \zeta_2$. This eliminates the variable z_p from the integral. We continue by contracting the z_{p-1}-contour, again with residue at $z_{p-1} = \zeta_2$, and so on until variable z_{k+1} is eliminated. Next, we contract the z_1-contour and gain a residue at $z_1 = \zeta_1$. We keep doing so until we have contracted all contours except for the variables ζ_1, ζ_2 and z_k. We will also obtain a factor of $(-1)^{k-2}$ while eliminating variables $z_2, \ldots, z_{k-1}$ due to the factor $(\zeta_1 - z_2)\cdots(z_{k-1} - z_k)$ in the integrand. Factoring out another -1 shows that

$$
\begin{aligned}
L^{\varepsilon^k}_{\vec{0}}(i,j) =& (-1)^{k-1} \oint\limits_{\gamma_{\tau_1}} d\zeta_1 \oint\limits_{\gamma_{\tau_2}} d\zeta_2 \oint\limits_{\gamma_{R_k}} dz_k \\
& \frac{G\big(z_k \,|\, \Delta_k n, \Delta_k m, \Delta_k a\big)\, G\big(\zeta_1 \,|\, n_{k-1} - i, m_{k-1} - m(i), a_{k-1} - a(i)\big) \left(\frac{1-\zeta_1}{1-z_1}\right)^{\mathbf{1}_{\{k=1\}}}}{G\big(\zeta_2 \,|\, n_k - j + 1, m_k - m(j), a_k - a(j)\big)\, (z_k - \zeta_1)\, (z_k - \zeta_2)}.
\end{aligned}
$$

Finally, we eliminate the z_k-contour and gain a residue at $z_k = \zeta_1$ followed by one at $z_k = \zeta_2$ (recall $\tau_1 > \tau_2$). This gives the difference $(1-\sqrt{q})\big(L_k(i,j) - L_{k-1}(i,j)\big)$. ∎

We remark that the identity matrix in the Fredholm determinantal representation for $L(i,j\,|\,\theta)$ will appear from the sum $\sum_k \theta^{\varepsilon^k}(i) L^{\varepsilon^k}_{\vec{0}}(i,j)$ by way of Lemma 4.5.

Lemma 4.6. *Consider* $0 \leqslant k_1 < k_2 \leqslant p$ *and*

$$\vec{\varepsilon} = (2, \ldots, 2, \varepsilon_{\max\{k_1,1\}}, \ldots, \varepsilon_{\max\{k_2,p-1\}}, 1, \ldots, 1) \in \{1,2\}^{p-1}.$$

Suppose $\vec{\delta}$ *equals 1 on indices over the interval* $(k_1, k_2]$ *and 0 elsewhere. Then,* $L^{\vec{\varepsilon}}_{\vec{\delta}} = (-1)^{k_1}(1-\sqrt{q})L^{\vec{\varepsilon}}_{(k_1,k_2]}$*. Furthermore,* $L^{\vec{\varepsilon}}_{(p-1,p]}$ *equals* L_p *where*

$$L_p(i,j) = \frac{\mathbf{1}_{\{i>n_{p-1}\}}}{1-\sqrt{q}} \oint\limits_{\gamma_{\tau_2}} d\zeta_2 \oint_{\gamma_{R_p}(1)} dz_p \, \frac{G\big(z_p \,|\, n_p - i, \Delta_p m, \Delta_p a\big)}{G\big(\zeta_2 \,|\, n_p - j + 1, m_p - m(j), a_p - a(j)\big)\,(z_p - \zeta_2)}.$$

Proof. By Lemma 4.4, $L^{\vec{\varepsilon}}_{\vec{\delta}} = 0$ unless $\vec{\varepsilon}$ is as given in the statement of this lemma. Consider again the contour integral presentation of $L^{\vec{\varepsilon}}_{\vec{\delta}}(i,j)$ from Lemma 4.2. The contours around 0 are those in variables $z_1, \ldots, z_{k_1}$ and $z_{k_2+1}, \ldots, z_p$. We also have $R_1 < \cdots < R_{k_1}$ and $R_p < \cdots < R_{k_2+1}$.

As in the proof of the previous lemma, we contract the contours around 0, gaining residues, and present $L^{\vec{\varepsilon}}_{\vec{\delta}}(i,j)$ as an integral involving variables $\zeta_1, \zeta_2, z_{k_1+1}, \ldots, z_{k_2}$. The calculation of this is straightforward and we omit the details. The reason a factor $\big(\frac{1-\zeta_1}{1-z_1}\big)^{\mathbf{1}_{\{k_1=0\}}}$ appears is that when $k_1 = 0$ the z_1-contour is not contracted, so no residue is obtained at $z_1 = \zeta_1$.

The final result is a presentation of $L^{\vec{\varepsilon}}_{\vec{\delta}}(i,j)$ that appears like $(1-\sqrt{q})L^{\vec{\varepsilon}}_{(k_1,k_2]}(i,j)$ from Definition 4.2 except the indicator $\mathbf{1}_{\{i>n_{k_1}, j \leqslant n_{k_2}\}}$ is absent. To see why we may assume $i > n_{k_1}$, observe the variable ζ_1 appears in the integrand of $L^{\vec{\varepsilon}}_{(k_1,k_2]}(i,j)$ as

$$\frac{G(\zeta_1 \,|\, n_{k_1} - i, m_{k_1} - m(i), a_{k_1} - a(i))}{z_{k_1+1} - \zeta_1}.$$

When $n_{k_1} - i \geqslant 0$, there is no pole in the ζ_1 variable inside γ_{τ_1} and the contour may be contracted to 0. Similarly, if $j > n_{k_2}$, there is no pole in ζ_2 inside γ_{τ_2}.

To simplify $L^{\vec{\varepsilon}}_{(p-1,p]}$ note that it does not depend on $\vec{\varepsilon}$ as there is a single contour around 1 (the z_p-contour). Since $i > n_{p-1}$, its integrand decays at least to the order ζ_1^{-2} in the ζ_1 variable (the dependence is displayed above). Further, $m(i) = m_{p-1}$ and $a(i) = a_{p-1}$. So there are no poles at $\zeta_1 = 1-q$ and 1, and the ζ_1-contour can be contracted to ∞. In doing so we gain a residue at $z_1 = z_p$ whose value is $G(z_p \,|\, n_{p-1} - i, 0, 0)$. Then simplifying the integrand using the group property gives the desired expression for $L^{\vec{\varepsilon}}_{(p-1,p]}$. ∎

The following simplifies $L^{\vec{\varepsilon}}_{(k_1,k_2]}$ when $k_2 < p$.

Lemma 4.7. *If* $0 \leqslant k_1 < k_2 < p$ *and* $\vec{\varepsilon} \in \{1,2\}^{p-1}$ *then*

$$L^{\vec{\varepsilon}}_{(k_1,k_2]}(i,j) = \mathbf{1}_{\{j \leqslant n_{k_2-1}\}} \, L^{\vec{\varepsilon}}_{(k_1,k_2]}(i,j) + \mathbf{1}_{\{i > n_{k_1}, j \in (n_{k_2-1}, n_{k_2}]\}} \, J^{\vec{\varepsilon}}_{(k_1,k_2]}(i,j),$$

where

$$J^{\vec{\varepsilon}}_{(k_1,k_2]}(i,j) = \frac{1}{1-\sqrt{q}} \oint\limits_{\gamma_{\tau_1}} d\zeta_1 \oint\limits_{\vec{\gamma}_{R^{\vec{\varepsilon}}}} dz_{k_1+1} \cdots dz_{k_2}$$

$$\frac{\prod_{k_1<k<k_2} G\big(z_k \,|\, \Delta_k(n,m,a)\big) G\big(z_{k_2} \,|\, j-1-n_{k_2-1}, \, \Delta_{k_2}(m,a)\big) \, \left(\frac{1-\zeta_1}{1-z_1}\right)^{\mathbf{1}_{\{k_1=0\}}}}{G\big(\zeta_1 \,|\, i - n_{k_1}, m(i) - m_{k_1}, a(i) - a_{k_1}\big) \prod_{k_1<k<k_2} (z_k - z_{k+1}) \, (z_{k_1+1} - \zeta_1)}.$$

The contours in $J^{\vec{\varepsilon}}_{(k_1,k_2]}$ *are arranged like those in* $L^{\vec{\varepsilon}}_{(k_1,k_2]}$*.*

Proof. Consider $L^{\vec{\varepsilon}}_{(k_1,k_2]}(i,j)$ when $j \in (n_{k_2-1}, n_{k_2}]$. Since $k_2 < p$, we have $m(j) = m_{k_2}$ and $a(j) = a_{k_2}$. Therefore, the integrand depends on ζ_2 according to the term $G\big(\zeta_2 \,|\, n_{k_2} - j + 1, 0, 0\big)\,(z_{k_2} - \zeta_2)$ in the denominator. Since $n_{k_2} - j \geqslant 0$, we may contract the ζ_2-contour to infinity with residue at $\zeta_2 = z_{k_2}$ to find that

$$\oint_{\gamma_{\tau_2}} d\zeta_2 \, \frac{1}{G(\zeta_2 \,|\, n_{k_2} - j + 1, 0, 0)(z_{k_2} - \zeta_2)} = \frac{1}{G(z_{k_2} \,|\, n_{k_2} - j + 1, 0, 0)}.$$

So we evaluate the integral in ζ_2 and simplify the integrand using the group property of G, which results in $J^{\vec{\varepsilon}}_{(k_1,k_2]}$. ■

We may now write $L(i,j\,|\,\theta)$ from (4.7) in the following way by using Definitions 4.1 and 4.2, as well as Lemmas 4.3, 4.4, 4.5 and 4.6. Observe that for $\vec{\varepsilon} = \varepsilon^k$ as in Lemma 4.5, $(-1)^{p-1+\sum_i \varepsilon_i^k + k - 1} = 1$. Also, for $0 \leqslant k_1 < k_2 \leqslant p$ and $\vec{\varepsilon}$ as in Lemma 4.6,

$$(-1)^{p-1+\sum_i \varepsilon_i + k_1} = (-1)^{k_1 + \min\{k_2, p-1\}} \cdot (-1)^{\varepsilon[k_1,k_2]}, \quad \text{where } (-1)^{\varepsilon[k_1,k_2]} \text{ is around (2.6).}$$

Putting all this together with (4.7) we find that

$$\begin{aligned} L(i,j\,|\,\theta) = {} & \sum_{k=1}^{p} c(i,j)\,\theta^{\varepsilon^k}(i)\,\big(L_k - L_{k-1}\big)(i,j) \; + \\ & \sum_{0 \leqslant k_1 < k_2 \leqslant p} \; \sum_{\substack{\vec{\varepsilon} \in \{1,2\}^{p-1} \\ \varepsilon_i = 2 \text{ if } i < \max\{k_1, 1\} \\ \varepsilon_i = 1 \text{ if } i > \min\{k_2, p-1\}}} (-1)^{\varepsilon[k_1,k_2] + k_1 + \min\{k_2, p-1\}}\, c(i,j)\,\theta^{\vec{\varepsilon}}(i)\, L^{\vec{\varepsilon}}_{(k_1,k_2]}(i,j)\,. \end{aligned} \tag{4.8}$$

It will be convenient to write the matrices associated to $L(i,j\,|\,\theta)$ from (4.8) in the $p \times p$ block form, which motivates the following definition.

Definition 4.3. $A(\theta)$ and $B(\theta)$ are $N \times N$ matrices with a $p \times p$ block form as follows. Recall Definitions 4.1 and 4.2, and notation introduced in §2.1. In particular, from (2.4), that the (r,s)-block of a matrix M is denoted $M(r,i;s,j)$, and that $r^* = \min\{r, p-1\}$.

(1) Define matrix $B(\boldsymbol{\theta})$, $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_{p-1})$, by

$$B(r,i;s,j\,|\,\boldsymbol{\theta}) = (1 + \Theta(r\,|\,s)) \cdot c(r,i;s,j) \cdot \mathbf{1}_{\{s < r^*\}} \, \frac{1}{1-\sqrt{q}} \oint_{\gamma_\tau} dw \, \frac{1}{G\big(w \,|\, i - j + 1,\, \Delta_{s,r^*}(m,a)\big)},$$

where the circular contour γ_τ around 0 had radius $\tau < 1 - \sqrt{q}$ and $\Theta(r\,|\,s)$ is given by (2.6).

(2) Define matrix $A(\boldsymbol{\theta}) = A_1(\boldsymbol{\theta}) + A_2(\boldsymbol{\theta})$ as follows.

$$A_1(r,i;s,j\,|\,\boldsymbol{\theta}) = \sum_{k=0}^{p} \Theta(r\,|\,k) \cdot L[k,k\,|\,\emptyset](r,i;s,j), \text{ where}$$

$$L[k,k\,|\,\emptyset]\,(r,i;s,j) = c(r,i;s,j)\,\mathbf{1}_{\{s<k<r^*\}} \cdot L_k(r,i;s,j).$$

Let $0 \leqslant k_1, k_2 \leqslant p$ and $\vec{\varepsilon} \in \{1,2\}^{p-1}$. Set

$$A_2(r,i;s,j \,|\, \theta) = \sum_{\substack{k_1<k_2,\,\vec{\varepsilon} \\ \varepsilon_k=2 \text{ if } k<\max\{k_1,1\} \\ \varepsilon_k=1 \text{ if } k>\min\{k_2,p-1\}}} (-1)^{\varepsilon[k_1,k_2]+k_1+k_2^*} \cdot \theta(r\,|\,\vec{\varepsilon}) \times \Big[L^{\vec{\varepsilon}}[k_1,k_2 \,|\, (k_1,k_2]] + L^{\vec{\varepsilon}}[k_1 \,|\, (k_1,k_2]] + \mathbf{1}_{\{k_1=p-1,k_2=p\}} L[p \,|\, p] \Big](r,i;s,j),$$

where recalling L_p and $J^{\vec{\varepsilon}}_{(k_1,k_2]}$ from Lemmas 4.6 and 4.7, respectively, we define

$$L^{\vec{\varepsilon}}[k_1,k_2 \,|\, (k_1,k_2]](r,i;s,j) = c(r,i;s,j)\, \mathbf{1}_{\{k_1<r^*,\ s^*<k_2,\, k_1<k_2\}} \cdot L^{\vec{\varepsilon}}_{(k_1,k_2]}(r,i;s,j).$$

$$L^{\vec{\varepsilon}}[k_1 \,|\, (k_1,k_2]](r,i;s,j) = c(r,i;s,j)\, \mathbf{1}_{\{k_1<r^*,\ s=k_2<p,\, k_1<k_2\}} \cdot J^{\vec{\varepsilon}}_{(k_1,k_2]}(r,i;s,j).$$

$$L[p \,|\, p](r,i;s,j) = c(r,i;s,j)\, \mathbf{1}_{\{r=p\}} \cdot L_p(r,i;s,j).$$

Some comments on these matrices. In terms of the $p \times p$ block structure, $B(\theta)$ is lower triangular with zeroes on the diagonal blocks. Its last two column blocks are zero as well. The matrix $A_1(\theta)$ is also strictly block-lower-triangular with the last three column blocks being zero. The matrix $L^{\vec{\varepsilon}}[k_1,k_2 \,|\, (k_1,k_2]]$ has non-zero blocks strictly above row k_1 ($r > k_1$) and at or below column k_2. The matrix $L^{\vec{\varepsilon}}[k_1 \,|\, (k_1,k_2]]$ has non-zero blocks only on column $k_2 < p$ and above row k_1. The matrix $L[p \,|\, p]$ has non-zero block only on row p.

Theorem 2. *Let* $\mathbf{G}$ *be the growth function defined by* (1.1). *Let* $A(\theta)$ *and* $B(\theta)$ *be from Definition 4.3, and suppose* $p \geqslant 2$. *For* $m_1 < m_2 < \cdots < m_p$ *and* $n_1 < n_2 < \cdots < n_p$, *we have*

$$\mathbf{Pr}\left[\mathbf{G}(m_1,n_1) < a_1, \mathbf{G}(m_2,n_2) < a_2, \ldots, \mathbf{G}(m_p,n_p) < a_p\right] = \oint_{\gamma_r^{p-1}} d\theta_1 \cdots d\theta_{p-1} \frac{1}{\prod_{k=1}^{p-1}(\theta_k - 1)} \det\left(I + A(\theta) + B(\theta)\right).$$

Here, $\gamma_r^{p-1} = \gamma_r \times \cdots \times \gamma_r$ ($p-1$ *times) and* γ_r *is a counter-clockwise, circular contour around the origin of radius* $r > 1$.

In order to prove the theorem we need the following.

Lemma 4.8. *Set, for* $0 < \tau < 1 - \sqrt{q}$,

$$B(i,j) = \frac{1}{1-\sqrt{q}} \oint_{\gamma_\tau} dw \, \frac{1}{G\big(w \,|\, i-j+1,\ m(i)-m(j),\ a(i)-a(j)\big)}.$$

Then,

$$\begin{aligned}(L_k - L_{k-1})(i,j) &= \mathbf{1}\{\, i,j \in (n_{k-1}, n_k] \,\} \cdot \mathbf{1}\{i=j\} \ + \\ &\quad \mathbf{1}\{\, i \in (n_{k-1}, n_k],\ j \leqslant n_{\min\{k-1,p-2\}} \,\} \cdot B(i,j) \ + \\ &\quad \mathbf{1}\{\, i > n_k,\ j \leqslant n_k,\ k \leqslant p-2 \,\} \cdot L_k(i,j) - \mathbf{1}\{\, i > n_{k-1},\ j \leqslant n_{k-1},\ k \leqslant p-1 \,\} \cdot L_{k-1}(i,j).\end{aligned}$$

Proof. Recall from Definition 4.1:

$$L_k(i,j) = \frac{1}{1-\sqrt{q}} \oint_{\gamma_{\tau_1}} d\zeta_1 \oint_{\gamma_{\tau_2}} d\zeta_2 \, \frac{G\big(\zeta_1 \,|\, n_k - i, m_k - m(i), a_k - a(i)\big)}{G\big(\zeta_2 \,|\, n_k - j + 1, m_k - m(j), a_k - a(j)\big)\,(\zeta_1 - \zeta_2)}.$$

- If $j > n_k$ then there is no pole at $\zeta_2 = 0$ in the above and we can contract the ζ_2-contour to 0. So $L_k(i,j) = 0$, which means $L_k(i,j) = \mathbf{1}_{\{j \leqslant n_k\}} L_k(i,j)$.

- If $i > n_k$ and $m(i) = m_k$ (so $a(i) = a_k$ as well), then $L_k(i,j) = 0$ because the ζ_1-contour may be contracted to ∞. The condition $i > n_k$ and $m(i) = m_k$ is the same as $i > n_k$ and $k \geqslant p-1$. Indeed, if $i > n_k$ and $k \geqslant p-1$ then $m(i) = m_k = m_{p-1}$ ($i > n_p$ is vacuous). Therefore, $L_k(i,j) = \mathbf{1}_{\{i \leqslant n_k, j \leqslant n_k\}} L_k(i,j) + \mathbf{1}_{\{i > n_k, j \leqslant n_k, k \leqslant p-2\}} L_k(i,j)$.

- When $i \leqslant n_k$ we can contract the ζ_1-contour to 0, picking up a residue at $\zeta_1 = \zeta_2$, which equals $B(i,j)$. Also, $B(i,j) = 0$ if $j > i$ because there is no pole at $w = 0$ in that case. Consequently,
$$L_k(i,j) = \mathbf{1}_{\{i \leqslant n_k, j \leqslant n_k, j \leqslant i\}} B(i,j) + \mathbf{1}_{\{i > n_k, j \leqslant n_k, k \leqslant p-2\}} L_k(i,j).$$

- If $m(i) = m(j)$ then
$$B(i,j) = (1-\sqrt{q})^{i-j} \oint_{\gamma_\tau} d\zeta\, \zeta^{j-i-1} = \mathbf{1}_{\{i=j\}}.$$

Putting all this together we infer that
$$\begin{aligned} L_k(i,j) = {} & \mathbf{1}\{\, i \leqslant n_k,\ j \leqslant n_k,\ i = j\,\} \ + \\ & \mathbf{1}\{\, i \leqslant n_k,\ j \leqslant n_k,\ j \leqslant i,\ m(i) \neq m(j)\,\} \cdot B(i,j) \ + \\ & \mathbf{1}\{\, i > n_k,\ j \leqslant n_k,\ k \leqslant p-2\,\} \cdot L_k(i,j) \end{aligned}$$
Taking the difference of $L_k(i,j)$ from $L_{k-1}(i,j)$ by using the expression above gives the expression in the lemma, except that the indicator in front of $B(i,j)$ reads $i \in (n_{k-1}, n_k]$, $j \leqslant n_{k-1}$ and $m(i) \neq m(j)$. However, when $j \leqslant n_{k-1}$, the condition $m(i) \neq m(j)$ is precisely $j \leqslant n_{\min\{k-1,p-2\}}$. ■

Proof of Theorem 2. We have the basic integral expression for the multi-point probability from Lemma 3.3. The matrix $L(i,j \,|\, \theta)$ is given by (4.8), which we will prove to equal $I + A(\theta) + B(\theta)$.

The matrix $A_2(\theta)$ is the one written in the second line of equation (4.8). We should explain the conditions $k_1 < \min\{r, p-1\}$ and $\min\{s, p-1\} < k_2$ in $L^{\vec{\varepsilon}}[k_1,k_2 \,|\, (k_1,k_2]]$. Also, why is it that $k_1 < \min\{r, p-1\}$ and $s = k_2 < p$ in $L^{\vec{\varepsilon}}[k_1 \,|\, (k_1,k_2]]$.

The condition $k_1 < r$ appears because in the definition of $L^{\vec{\varepsilon}}_{(k_1,k_2]}(i,j)$ we have $i < n_{k_1}$, while we know $i \in (n_{r-1}, n_r]$. The condition $k_1 < p-1$ appears because $L^{\vec{\varepsilon}}_{(k_1,k_2]}$ is zero if $k_1 \geqslant p-1$ by Lemma 4.6. The condition on s arises from the decomposition of $L^{\vec{\varepsilon}}_{(k_1,k_2]}$ in Lemma 4.7. Since $j \in (n_{s-1}, n_s]$, we have $s \leqslant k_2$, which we decompose into two conditions: (a) $\mathbf{1}_{\{s \leqslant k_2, k_2 = p\}} + \mathbf{1}_{\{s < k_2, k_2 < p\}} = \mathbf{1}_{\{\min\{s,p-1\} < k_2\}}$ and (b) $\mathbf{1}_{\{s = k_2 < p\}}$. In case (b) the matrix $L^{\vec{\varepsilon}}_{(k_1,k_2]}$ becomes $J^{\vec{\varepsilon}}_{(k_1,k_2]}$ by Lemma 4.7, and this results in the matrix $L^{\vec{\varepsilon}}[k_1 \,|\, (k_1,k_2]]$.

We have to show that the matrix associated to the first line in (4.8) equals $I + A_1(\boldsymbol{\theta}) + B(\boldsymbol{\theta})$. If we write the statement of Lemma 4.8 in block notation, it reads

$$(L_k - L_{k-1})(r,i;s,j) = \mathbf{1}_{\{r=k=s\}} \cdot \mathbf{1}_{\{i=j\}} + \mathbf{1}_{\{r=k,\, s+1 \leqslant \min\{r,p-1\}\}} \cdot B(r,i;s,j) + \mathbf{1}_{\{r>k,\, s\leqslant k,\, k\leqslant p-2\}} \cdot L_k(r,i;s,j) - \mathbf{1}_{\{r>k-1,\, s\leqslant k-1,\, k\leqslant p-1\}} \cdot L_{k-1}(r,i;s,j). \tag{4.9}$$

We need to consider the weighted sum $\sum_k \theta^{\varepsilon^k}(i) \cdot c(r,i;s,j) \times$ (4.9).

Observe that if $i \in (n_{k-1}, n_k]$ then

$$\theta^{\varepsilon^k}(i) = \theta(k \,|\, \varepsilon^k) = \theta_1^{-\mathbf{1}_{\{i\leqslant n_1\}}} \cdots \theta_{k-1}^{-\mathbf{1}_{\{i\leqslant n_{k-1}\}}} \, \theta_k^{\mathbf{1}_{\{i>n_k\}}} \cdots \theta_{p-1}^{\mathbf{1}_{\{i>n_{p-1}\}}} = 1.$$

Therefore, summing $\theta^{\varepsilon^k}(i)\mathbf{1}_{\{r=k=s\}}\mathbf{1}_{\{i=j\}}$ over k and multiplying by $c(r,i;s,j)$ gives the matrix $\mathbf{1}_{\{i=j\}}c(r,i;s,j)$, which is the identity since $c(r,i;s,j)$ is a conjugation factor.

Consider the third term on the r.h.s. of (4.9) containing the difference between L_k and L_{k-1}. This term is zero unless $s < r$, and k satisfies $s \leqslant k \leqslant r$. When $s < k < r$, it equals $\mathbf{1}_{\{k<p-1\}}(L_k - L_{k-1})(r,i;s,j)$. Also, the condition $s < k < r$ is vacuous unless $s < r-1$. When $k = s$, the term becomes $\mathbf{1}_{\{s<p-1\}}L_s(r,i;s,j)$. When $k = r$, it equals $\mathbf{1}_{\{r<p\}}L_{r-1}(r,i;s,j)$. We will see in the following paragraph that $L_s(r,i;s,j) = B(r,i;s,j)$. Thus, we find appearances of $B(r,i;s,j)$ in the third term from L_k when $k = s$, and from L_{k-1} when $k = s+1$. Accounting for these $B(r,i;s,j)$, we find the weighted sum

$$\sum_k \theta^{\varepsilon^k}(i)\big(\text{third term of (4.9)}\big) = (\mathrm{I}) + (\mathrm{II}), \quad \text{where}$$

$$(\mathrm{I}) = \mathbf{1}_{\{s<r,\, s<p-1\}}\Big(\theta(r \,|\, \varepsilon^s) - \big(\mathbf{1}_{\{s+1<r,\, s+1<p-1\}} + \mathbf{1}_{\{s+1=r,\, r<p\}}\big)\theta(r \,|\, \varepsilon^{s+1})\Big)B(r,i;s,j),$$

$$(\mathrm{II}) = \mathbf{1}_{\{s+1<r\}}\Big(\sum_{k:\, s+1<k<r,\, k<p-1} \theta(r \,|\, \varepsilon^k)\big(L_k - L_{k-1}\big)(r,i;s,j) + \mathbf{1}_{\{s<p-2\}}\, \theta(r \,|\, \varepsilon^{s+1})L_{s+1}(r,i;s,j) - \mathbf{1}_{\{r<p\}}\, L_{r-1}(r,i;s,j)\Big).$$

We have used that $\theta^{\varepsilon^k}(i) = \theta(r \,|\, \varepsilon^k)$.

Consider term (I). If $s < r$ and $s < p-1$ then

$$\mathbf{1}_{\{s+1<r,\, s+1<p-1\}} + \mathbf{1}_{\{s+1=r,\, r<p\}} = 1 - \mathbf{1}_{\{r=p,\, s=p-2\}},$$

which gives the coefficient $\Theta(r \,|\, s)$ in term (I) if we recall its definition from (2.6). If we take this contribution of $\mathbf{1}_{\{s<r,\, s<p-1\}}\Theta(r \,|\, s)B(r,i;s,j)$, and combine it with

$$\sum_k \theta^{\varepsilon^k}(i)\, \mathbf{1}_{\{r=k,\, s<r,\, s<p-1\}}B(r,i;s,j) = \mathbf{1}_{\{s<\min\{r,p-1\}\}}\, B(r,i;s,j)$$

coming from the k-summation of the second term of (4.9), then, after conjugation by $c(r,i;s,j)$, we get the matrix $B(\boldsymbol{\theta})$ from Definition 4.3.

Now consider term (II). If we express it as a sum involving the $L_k(r,i;s,j)$ then the coefficient of $L_k(r,i;s,j)$ is $\mathbf{1}_{\{s<k<\min\{r,p-1\}\}} \cdot (\theta(r \,|\, \varepsilon^k) - \theta(r \,|\, \varepsilon^{k+1}))$. Recalling $\Theta(r \,|\, k)$, we see that $\theta(r \,|\, \varepsilon^k) - \theta(r \,|\, \varepsilon^{k+1}) = \Theta(r \,|\, k)$ because $s < p-2$, due to $s < k < \min\{r, p-1\}$. Hence, the contribution of L_k

appears as $\Theta(r\,|\,k)\,L_k(r,i;s,j)$. The sum over k followed by multiplication by $c(r,i;s,j)$ equals the matrix $A_1(\theta)$.

Finally, we show that $L_s(i,j) = B(i,j)$ for $j \in (n_{s-1}, n_s]$ and $s \leqslant p-2$ as is the case above. Indeed, we have $m(j) = m_s$ and $a(j) = a_s$, which means that

$$L_s(i,j) = \oint\limits_{\gamma_{\tau_1}} d\zeta_1 \oint\limits_{\gamma_{\tau_2}} d\zeta_2 \, \frac{G\big(\zeta_1\,|\,n_s - i, m_s - m(i), a_s - a(i)\big)}{G\big(\zeta_2\,|\,n_s - j + 1, 0, 0\big)\,(\zeta_1 - \zeta_2)}.$$

We can contract the ζ_2-contour to ∞, since $j \leqslant n_s$, but doing so leaves a residue at $\zeta_2 = \zeta_1$. Its value is $B(i,j)$. ■

4.3. **Distribution function of the single point law.** When $p = 1$ one can write a Fredholm determinantal expression for $\mathbf{Pr}\,[\mathbf{G}(m,n) < a]$ where the matrix is in terms of a double contour integral. Such formulas are nowadays frequent as discrete approximations to Tracy-Widom laws, so this section is meant to provide some intuition for our orthogonalization procedure.

From Lemma 3.1 we see that $\mathbf{Pr}\,[\mathbf{G}(m,n) < a] = \det\big(\nabla^{j-i-1} w_m(a)\big)_{n\times n}$. Consider the following matrix $B = [b_{kj}]$, which is a slight variant of B from (4.3).

$$b_{kj} = \oint\limits_{\gamma_\tau} d\zeta \, \frac{1}{G^*\big(\zeta\,|\,k - j + 1, m, a-1\big)}\,.$$

The radius $\tau < 1 - q$. The matrix is lower triangular with 1s on the diagonal, so $\det(B) = 1$. We have

$$\mathbf{Pr}\,[\mathbf{G}(m,n) < a] = \det\big(\ell_{ij}\big)\,, \quad \ell_{ij} = \sum_{k=1}^{N} (-1)^{k+i} \nabla^{k-i-1} w_m(a) b_{kj}\,.$$

Using (4.2) and Lemma 4.1 we find that

$$\ell_{ij} = \oint\limits_{\gamma_\tau} d\zeta \oint\limits_{\gamma_R} dz \, \frac{G^*(z\,|\,n-i, m, a-1)}{G^*(\zeta\,|\,n-j+1, m, a-1)(z-\zeta)}\,.$$

The radii $\tau < 1 - q$ and $R > 1$. By collecting residue of the z-integral at $z = \zeta$, we infer that

$$\begin{aligned}\ell_{ij} &= \oint\limits_{\gamma_\tau} d\zeta\, \zeta^{j-i-1} + \oint\limits_{\gamma_\tau} d\zeta \oint\limits_{\gamma_r(1)} dz \, \frac{G^*(z\,|\,n-i, m, a-1)}{G^*(\zeta\,|\,n-j+1, m, a-1)(z-\zeta)} \\ &= \mathbf{1}_{\{i=j\}} + M(i,j).\end{aligned}$$

Now we arrange the radii to have $\tau < 1 - \sqrt{q} < 1 - r < 1 - q$.

If we write $i = \lfloor c_0 n^{1/3} u \rfloor$ and $j = \lceil c_0 n^{1/3} v \rceil$, then a direct asymptotical analysis of $M(i,j)$ leads to the Airy kernel (2.16) under KPZ scaling.

5. Asymptotics: formulation in the KPZ-scaling limit

In order to prove Theorem 1 we will consider the limit of the determinantal expression from Theorem 2 under KPZ scaling. We will do so in several steps. In §5.1 we define the Hilbert space where all matrices are embedded in the pre and post limit. The proof of convergence of the determinant will be based on a steepest descent analysis of the matrix entries. In §5.2 we provide contours of descent and behaviour of the entries around critical points. The proof of convergence is in §5.3. There is a technical addendum in §5.4, where it is also proved that the limit from Theorem 1 is a probability distribution.

5.1. **Setting for asymptotics.** Consider the space $X = \overbrace{\mathbb{R}_{<0} \oplus \cdots \oplus \mathbb{R}_{<0}}^{p-1} \oplus \mathbb{R}_{>0}$ and a measure λ on it defined by $\int_X d\lambda\, f = \sum_{k=1}^{p-1} \int_{-\infty}^{0} dx\, f(k,x) \;+\; \int_0^\infty dx\, f(p,x)$. Define the Hilbert space

$$\mathcal{H} = L^2(X,\lambda) \cong \underbrace{L^2(\mathbb{R}_{<0}, dx) \oplus \cdots \oplus L^2(\mathbb{R}_{<0}, dx)}_{p-1} \oplus L^2(\mathbb{R}_{>0}, dx)\,. \tag{5.1}$$

Recall the partition $\{1\,\ldots,N\} = (0,n_1] \cup \cdots (n_{p-1},n_p]$. Embed indices from $\{1,\ldots,N\}$ into X by mapping each index i into a unit length interval in the following manner.

$$i \mapsto \begin{cases} \text{points } (k,u) & \text{for } \; i-1 < n_k + u \leqslant i \; \text{ if } i \in (n_{k-1}, n_k] \text{ and } k < p, \\ \text{points } (p,u) & \text{for } \; i-1 < n_{p-1} + u \leqslant i \; \text{ if } i \in (n_{p-1}, n_p]. \end{cases} \tag{5.2}$$

Observe that for $k < p$ the block $(n_{k-1}, n_k]$ is mapped to the interval $(-\Delta_k n, 0]$ and for $k = p$ it is mapped to $(0, \Delta_p n]$.

An $N \times N$ matrix M embeds as a kernel $\widetilde{M}$ on $\mathcal{H}$ by

$$\widetilde{M}(r,u;\, s,v) = M\big(r,\, n_{\min\{r,p-1\}} + \lceil u \rceil;\; s,\, n_{\min\{s,p-1\}} + \lceil v \rceil\big). \tag{5.3}$$

Here we have used the block notation (2.4) and $\lceil u \rceil$ is the integer part of u after rounding up. The range of u and v lie in the aforementioned intervals determined by each block, but we may extend it to all of $\mathbb{R}_{<0}$ (and to $\mathbb{R}_{>0}$ for the final blocks) by making $\widetilde{M}$ zero. Then, by design,

$$\det\left(I + \widetilde{M}\right)_{\mathcal{H}} = \det\left(I + M\right)_{N \times N}$$

where

$$\det\left(I + \widetilde{M}\right)_{\mathcal{H}} = 1 + \sum_{k \geqslant 1} \frac{1}{k!} \int_{X^k} d\lambda(r_1,u_1) \cdots d\lambda(r_k,u_k) \det\left(\widetilde{M}(r_i,u_i;\, r_j,u_j)\right)_{k \times k}.$$

This is because $\widetilde{M}$ is constant to $M(i,j)$ on a square of the form $[\tilde{i}-1,\tilde{i}) \times [\tilde{j}-1,\tilde{j})$ determined according to the correspondence (5.2), and zero elsewhere.

In order to perform asymptotics we should rescale variables of $\widetilde{M}$ according to KPZ scaling (1.3). In this regard, recalling $\nu_T = c_0 T^{1/3}$, we change variables $(r,u) \mapsto (r, \nu_T \cdot u)$ in the Fredholm determinant of $\widetilde{M}$ above. So if we define a new matrix kernel

$$F(r,u;\, s,v) = \nu_T\, \widetilde{M}\big(r,\, \nu_T \cdot u;\; s,\, \nu_T \cdot v\big), \tag{5.4}$$

then

$$\det\left(I + F\right)_{\mathcal{H}} = \det\left(I + M\right)_{N \times N}.$$

We will use the following estimate about Fredholm determinants.

Lemma 5.1. *Let* A *and* E *be matrix kernels over a space* $L^2(X,\mu)$*, which satisfy the following for some positive constants* C_1*,* C_2 *and* $\eta \leqslant 1$*. There are non-negative functions* $a_1(x), a_2(x), e_1(x), e_2(x)$ *on* X *such that*

$$|A(x,y)| \leqslant a_1(x)a_2(y) \quad \textit{and} \quad |E(x,y)| \leqslant \eta\, e_1(x)e_2(y).$$

Moreover, both $a_1(x), e_1(x) \leqslant C_1$ *and both* $\int_X d\mu(x)\, a_2(x)$*,* $\int_X d\mu(x)\, e_2(x) \leqslant C_2$*. Then there is a constant* $C_3 = C_3(C_1, C_2)$ *such that*

$$\left|\det(I + A + E)_{L^2(X,\mu)} - \det(I + A)_{L^2(X,\mu)}\right| \leqslant \eta\, C_3.$$

Proof. For $x_1, \ldots, x_k \in X$, consider the determinant of $[A(x_i, x_j) + E(x_i, x_j)]$. Using multi-linearity, Hadamard's inequality, and the bounds on $a_1(x)$ and $e_1(x)$, we find that

$$\left|\det\left(A(x_i,x_j) + E(x_i,x_j)\right) - \det\left(A(x_i,x_j)\right)\right| \leqslant \sum_{S \subset [k],\, S \neq \emptyset} \eta^{|S|} k^{k/2} C_1^k \prod_{j \in S} e_2(x_j) \prod_{j \notin S} a_2(x_j).$$

If we integrate the above over every x_j, use the bound on the integrals of $a_2(x)$ and $e_2(x)$, and then collect contributions of η, we see that

$$\int_{X^k} d\mu(x_1) \cdots d\mu(x_k)\ \left|\det\left(A(x_i,x_j) + E(x_i,x_j)\right) - \det\left(A(x_i,x_j)\right)\right| \leqslant k^{k/2}(C_1C_2)^k((1+\eta)^k - 1).$$

Since $0 \leqslant \eta \leqslant 1$ we have that $(1+\eta)^k - 1 \leqslant \eta 2^k$. Consequently,

$$\left|\det(I + A + E)_{L^2(X,\mu)} - \det(I + A)_{L^2(X,\mu)}\right| \leqslant \eta \sum_{k \geqslant 1} \frac{k^{k/2}}{k!}(2C_1C_2)^k =: \eta C_3. \quad \blacksquare$$

We will use the following nomenclature for matrix kernels in the proof of convergence.

Definition 5.1. Let $M_1, M_2, \ldots$, be a sequence of matrices where M_N is an $N \times N$ matrix understood in terms of the $p \times p$ block structure above. Let $\widetilde{M}_N$ be the embedding of M_N into $\mathcal{H}$ as in (5.3), and F_N the rescaling according to (5.4).

- The matrices are *good* if there are non-negative, bounded and integrable functions $g_1(x)$, $\ldots, g_p(x)$ on $\mathbb{R}$ such that following holds. For every N,

$$|F_N(r; u,\, s, v)| \leqslant g_r(u)g_s(v) \quad \text{for every } \ 1 \leqslant r, s \leqslant p \ \text{ and } \ u, v \in \mathbb{R}.$$

- The matrices are *convergent* if there is a matrix kernel F on $\mathcal{H}$ such that the following holds uniformly in u, v restricted to compact subsets of $\mathbb{R}$.

$$\lim_{N \to \infty} F_N(r, u;\, s, v) = F(r, u;\, s, v) \quad \text{for every } \ 1 \leqslant r, s \leqslant p.$$

- The matrices are *small* if there is a sequence $\eta_N \to 0$ and functions $g_1, \ldots, g_p$ as for good matrices such the the following holds.

$$|F_N(r; u,\, s, v)| \leqslant \eta_N\, g_r(u)g_s(v) \quad \text{for every } \ 1 \leqslant r, s \leqslant p \ \text{ and } \ u, v \in \mathbb{R}.$$

Remark in the above definition that u and v will be negative or positive depending on the blocks, and we can think of F_N being zero outside the stipulated domain. It will be convenient to hide dependence of parameter N when discussing matrices and call a matrix good/convergent/small with N understood implicitly. The following are straightforward consequences of the definitions, dominated convergence theorem and Lemma 5.1.

(1) If $M_1, M_2, \dots$ are good and convergent matrices with limit F on $\mathcal{H}$ then

$$\det(I + F_N)_{\mathcal{H}} \to \det(I + F)_{\mathcal{H}} < \infty.$$

F satisfies the same goodness bound as its approximants.

(2) If $M_1, M_2, \dots$ are good and $S_1, S_2, \dots$ are small then

$$\det\left(I + F_{M_N} + F_{S_N}\right)_{\mathcal{H}} - \det\left(I + F_{M_N}\right)_{\mathcal{H}} \to 0,$$

where F_{M_N} is the rescaling of M_N according to (5.4) and similarly for F_{S_N}.

5.2. **Preparation.** In order to apply the method of steepest descent to the determinant from Theorem 2, we have to identify the limit of matrix kernels and also establish some decay estimates for them at infinity, so that the series expansion of the Fredholm determinant converges. To do this we need three things regarding the function $G(w \,|\, n, m, a)$.

First, we need to understand the asymptotic behaviour of $G(w \,|\, n, m, a)$ locally around its critical point under KPZ scaling of n, m, a. This is the content of Lemma 5.2. Second, we have to find descent contours for γ_τ and $\gamma_R(1)$ that appear in the description of $A(\theta)$ and $B(\theta)$. These are provided by Definition 5.2. Third, we have to establish decay of G along these contours, which is the subject of Lemma 5.3.

Recall $G(w \,|\, n, m, a)$ from (2.9) with the indices scaled as

$$\begin{aligned} n &= K - c_1 x K^{2/3} + c_0 \nu K^{1/3}, \\ m &= K + c_1 x K^{2/3}, \\ a &= c_2 K + c_3 \xi K^{1/3}. \end{aligned} \tag{5.5}$$

The constants c_i are given by (2.1). When $n = m$ and $a = c_2 n$ the function

$$\log G(w \,|\, n, m, a) = n \log w + (m + a)\log(1 - w) - m \log\left(1 - \frac{w}{1 - q}\right) - \log(G^*(1 - \sqrt{q} \,|\, n, m, a))$$

has a double critical point at

$$w_c = 1 - \sqrt{q}. \tag{5.6}$$

Lemma 5.2. *Assume that we have the scaling* (5.5) *and that* $|x|, |\xi|, |\nu| \leqslant L$ *for a fixed* L. *Then uniformly in* x, ξ, ν *and* $w \in \mathbb{C}$ *restricted to compact subsets,*

$$\lim_{K \to \infty} G\left(w_c + \frac{c_4 \cdot w}{K^{1/3}} \,\Big|\, n, m, a\right) = \mathcal{G}(w \,|\, 1, x, \xi - \nu) = \exp\left\{\frac{1}{3}w^3 + x w^2 - (\xi - \nu) w\right\}, \tag{5.7}$$

where

$$c_4 = \frac{q^{1/3}(1 - \sqrt{q})}{(1 + \sqrt{q})^{1/3}} = \frac{w_c}{c_0}. \tag{5.8}$$

The lemma is proved in Lemma 5.3 of [22] by considering the Taylor expansion of $\log G$ with the scaling (1.3).

The circular contours γ around 0 and 1 will be chosen according to the following two contours with appropriate values for the parameters.

Definition 5.2. Let $K > 0$ and $0 < d < K^{1/3}$. For $|\sigma| \leqslant \pi K^{1/3}$, set

$$w_0(\sigma) = w_0(\sigma; d) = w_c(1 - \frac{d}{K^{1/3}})e^{i\sigma K^{-1/3}} \tag{5.9}$$

and

$$w_1(\sigma) = w_1(\sigma; d) = 1 - \sqrt{q}(1 - \frac{d}{K^{1/3}})e^{i\sigma K^{-1/3}}. \tag{5.10}$$

Thus, w_0 is a circle around the origin of radius $w_c(1 - \frac{d}{K^{1/3}})$ and w_1 is a circle around 1 of radius $\sqrt{q}(1 - \frac{d}{K^{1/3}})$.

Recall the notation $(\nu)_+ = \max\{\nu, 0\}$ and $(\nu)_- = \max\{-\nu, 0\}$.

Lemma 5.3. *Assume* $|x|, |\xi| \leqslant L$ *for some fixed* $L > 0$. *Consider the scaling* (5.5) *where* ν *is such that* $n \geqslant 0$. *There are positive constants* $C_0, C_1, C_2, C_3, C_4, C_5$ *that depend on* q *and* L *such that the following holds. Let* $0 < \delta \leqslant C_0$. *There are positive constants* μ_1 *and* μ_2 *that depend on* q, L, δ *with the following property. If* $K \geqslant C_5$, *there is a choice of* $d = d(\nu)$ *such that*

$$\left|G(w_0(\sigma; d(\nu)) \,|\, n, m, a)\right|^{-1} \leqslant C_3 e^{-C_4\sigma^2 - \mu_1(\nu)_-^{3/2} + \mu_2(\nu)_+} \tag{5.11}$$

and

$$\left|G(w_1(\sigma; d(\nu)) \,|\, n, m, a)\right| \leqslant C_3 e^{-C_4\sigma^2 - \mu_1(\nu)_-^{3/2} + \mu_2(\nu)_+} \tag{5.12}$$

for every $|\sigma| \leqslant \pi K^{1/3}$. *If* $\nu \geqslant 0$ *then* $d(\nu)$ *may be any point in the interval* $[C_1, C_2K^{1/3}]$ *(*$C_1 < C_2 < 1$*). If* $\nu < 0$ *then* $d(\nu)$ *may be any point in the interval* $[C_1 + \delta \cdot (\nu)_-^{1/2}, C_2K^{1/3}]$.

The lemma is proved in combination of Lemmas 5.6 and 5.7 in [22]. It is based on a direct critical point analysis of the real parts of $\log G(w_0(\sigma, d))$ and $\log G(w_1(\sigma, d))$ with the scaling (1.3).

Now we mention the choice of conjugation constant μ from (2.12). During asymptotic analysis we have to set μ and the parameter δ from Lemma 5.3 such that they satisfy the following bounds (in addition to $0 < \delta \leqslant C_0$).

$$\delta < C_2 c_0^{1/2} t_p^{-1/2} \cdot \min_k \{(\Delta_k t)^{1/2}\} \quad \text{and} \quad \mu > \mu_2 \cdot \max_k \{(\Delta_k t)^{-1/3}\}. \tag{5.13}$$

So long as $t_k, |x_k|, |\xi_k| \leqslant L$, these constraints depend only on q, L and $\min_k\{\Delta_k t\}$.

The goodness and smallness of matrices will be certified as follows. Write

$$\psi(x) = -\mu_1 \cdot (x)_-^{3/2} + \mu_2 \cdot (x)_+ \tag{5.14}$$

where μ_1 and μ_2 are according to Lemma 5.3 and δ is set to satisfy (5.13). (The parameters t_k, x_k and ξ_k from (1.3) are now fixed.) Suppose $\Delta \geqslant \min_k\{(\Delta_k t)^{1/3}\} > 0$ and μ is as in (5.13). Then,

(5.15)
$$(1)\ e^{-\mu x+\psi(x/\Delta)} \leqslant e^{\frac{4(\mu\Delta)^3}{27\mu_1^2}} \text{ for } x \in \mathbb{R}. \text{ So it is bounded.}$$
$$(2)\ \int_{-\infty}^{\infty} dx\, e^{-\mu x+\psi(x/\Delta)} = \int_{-\infty}^{0} e^{-\mu_1\cdot(x/\Delta)_-^{3/2}+\mu\cdot(x)_-} + \int_0^{\infty} dx\, e^{(\frac{\mu_2}{\Delta}-\mu)\cdot(x)_+} < \infty.$$
$$(3)\ e^{-\mu x+\psi(x/\Delta)} \to 0 \text{ as } x \to \pm\infty.$$
$$(4)\ \int_{-\infty}^{0} dx\, e^{\mu x+\psi(x/\Delta)} < \infty.$$

5.3. **Convergence of the determinant.** In order to prove Theorem 1 by using Theorem 2, it suffices to show there is uniform convergence of $\det(I+A(\theta)+B(\theta))$ to $\det(I-F(\theta))_{\mathcal{H}}$ in terms of θ over the integration contour γ_r^{p-1}. Parameter θ enters the matrices in terms of $\theta(r\,|\,\vec{\varepsilon})$ and $\Theta(r\,|\,k)$ from (2.5) and (2.6). These quantities will play no role in the asymptotical analysis as all estimates will involve the basic matrices $L[\cdots]$. So all error terms will be uniform in θ, and we may suppress θ from notation as convenient.

The matrix A is good and convergent but B is not. (Under KPZ scaling, entries of B converge to entries of the form $\mathrm{Ai}(v-u)$, which does not have finite Fredholm determinant). On the other hand, $B^{p-1} = 0$ because B is strictly block-lower-triangular with last two column blocks being zero. So $(I+B)^{-1} = I - B + B^2 + \cdots + (-1)^{p-2}B^{p-2}$. We may then consider instead the determinant of $I + A - AB + \cdots + (-1)^{p-2}AB^{p-2}$. These matrices turn out to be small from AB^2 onward, and the first 2 are good and convergent. These considerations motivate the following.

Since $\det(I-B) = 1$,
$$\det(I+A+B) = \det(I+A+B)\det(I-B) = \det\Big(I - B^2 + A - AB\Big).$$
We will see in Lemma 5.4 that $B^2 = B_1 - B_2$, where B_1 is good and convergent. Proposition 5.1 will prove that A is good and convergent. We will also find, from Proposition 5.2, that $AB = (AB)_g + (AB)_s$ with $(AB)_g$ being good and convergent while $(AB)_s$ is small. Thus, under KPZ scaling, as $T \to \infty$,
$$\det(I+A+B) \approx \det(I + B_2 + (A - (AB)_g - B_1)).$$
Proposition 5.3 will prove that $P = A - (AB)_g - B_1$ is such that PB_2 is small. So
$$\det(I+B_2+P) \approx \det(I+B_2+P+PB_2) = \det(I+P)\det(I+B_2).$$
The matrix B_2 is strictly block-lower-triangular due to B being such. So $\det(I+B_2) = 1$. This means that
$$\det(I+A+B) \approx \det(I+P),$$
and the latter determinant converges under KPZ scaling. The limit of P is precisely the matrix kernel F from (2.15). So we will have proved Theorem 1 after proving the upcoming lemmas and propositions.

Lemma 5.4. *The matrix* $B^2 = B_1 - B_2$*, where* B_1 *and* B_2 *are as follows. Recall* $w_c = 1-\sqrt{q}$, $r^* = \min\{r, p-1\}$ *and likewise for* s^*.

$$B_1(r,i;s,j) = \sum_{k=0}^{p} (1+\Theta(r\,|\,k)) \cdot (1+\Theta(k\,|\,s)) \cdot L[k,k\,|\,\emptyset]\,(r,i;s,j).$$

$$B_2(r,i;s,j = \sum_{k=0}^{p} (1+\Theta(r\,|\,k)) \cdot (1+\Theta(k\,|\,s)) \cdot (\mathcal{S}L)[k,k\,|\,\emptyset]\,(r,i;s,j).$$

The matrix $(\mathcal{S}L)$ *is given by*

$$(\mathcal{S}L)[k,k\,|\,\emptyset]\,(r,i;s,j) = \mathbf{1}_{\{s<k<r^*\}}\, c(r,i;s,j)\, \frac{1}{w_c} \oint_{\gamma_{\tau_1}} d\zeta_1 \oint_{\gamma_{\tau_2}} d\zeta_2$$
$$\frac{(\zeta_1-\zeta_2)^{-1}}{G\big(\zeta_1\,|\,i-n_{k-1},\, \Delta_{k,r^*}(m,a)\big)\, G\big(\zeta_2\,|\,n_{k-1}-j+1,\, \Delta_{s,k}(m,a)\big)}.$$

The matrix B_1 *is good and convergent in the KPZ scaling limit with limiting kernel on* $\mathcal{H}$ *given by*

$$F^{(0)}(r,u;s,v) = \sum_{k=0}^{p} (1+\Theta(r\,|\,k)) \cdot (1+\Theta(k\,|\,s)) \cdot F[k,k\,|\,\emptyset]\,(r,i;s,j).$$

(Recall Fs *from Definition* (2.1).*)*

Proposition 5.1. *The matrix* A *is good and convergent due to the following. Suppose* $0 \leqslant k_1 < k_2 \leqslant p$.

(1) *The matrix* $L^{\vec{\varepsilon}}[k_1,k_2\,|\,(k_1,k_2]]$ *is good and convergent with limit* $(-1)^{k_2-k_1}\, F^{\vec{\varepsilon}}[k_1,k_2\,|\,(k_1,k_2]]$.

(2) *The matrix* $L^{\vec{\varepsilon}}[k_1\,|\,(k_1,k_2]]$ *is good and convergent with limit* $(-1)^{k_2-k_1}\, F^{\vec{\varepsilon}}[k_1\,|\,(k_1,k_2]]$.

(3) *The matrix* $L[k,k\,|\,\emptyset]$ *is good and convergent with limit* $F[k,k\,|\,\emptyset]$.

(4) *The matrix* $L[p\,|\,p]$ *is good and convergent with limit* $-F[p\,|\,p]$.

Lemma 5.5. *Suppose* $0 \leqslant k_1 < k_2 < p$. *We have*

$$L^{\vec{\varepsilon}}[k_1,k_2\,|\,(k_1,k_2]] \cdot B = \sum_{k_3=0}^{p} \big(1+\Theta(k_3\,|\,s)\big)\Big[L^{\vec{\varepsilon}}[k_1,k_2,k_3\,|\,(k_1,k_2]] - (\mathcal{S}L)^{\vec{\varepsilon}}[k_1,k_2,k_3\,|\,(k_1,k_2]]\Big].$$

$$L^{\vec{\varepsilon}}[k_1,k_2,k_3\,|\,(k_1,k_2]]\,(r,i;s,j) = \mathbf{1}_{\{k_1<r^*,\, s<k_3<k_2\}}\, c(r,i;s,j) \times$$
$$\frac{1}{w_c} \oint_{\gamma_{\tau_1}} d\zeta_1 \oint_{\gamma_{\tau_2}} d\zeta_2 \oint_{\gamma_{\tau_3}} d\zeta_3 \oint_{\vec{\gamma}_{R^{\vec{\varepsilon}}}} dz_{k_1+1}\cdots dz_{k_2} \left(\frac{1-\zeta_1}{1-z_1}\right)^{\mathbf{1}_{\{k_1=0\}}} \times$$
$$\frac{\prod_{k_1<k\leqslant k_2} G\big(z_k\,|\,\Delta_k(n,m,a)\big) \prod_{k_1<k<k_2}(z_k-z_{k+1})^{-1}\,(z_{k_1+1}-\zeta_1)^{-1}(z_{k_2}-\zeta_2)^{-1}(\zeta_2-\zeta_3)^{-1}}{G\big(\zeta_1\,|\,i-n_{k_1}, \Delta_{k_1,r^*}(m,a)\big)\, G\big(\zeta_2\,|\,\Delta_{k_3,k_2}(n,m,a)\big)\, G\big(\zeta_3\,|\,n_{k_3}-j+1, \Delta_{s,k_3}(m,a)\big)}.$$

The contours are arranged such that $\tau_2 < \tau_1, \tau_3 < 1-\sqrt{q}$. *Also,* $\vec{\gamma}_{R^{\vec{\varepsilon}}} = \gamma_{R_{k_1+1}}(1) \times \cdots \times \gamma_{R_{k_2}}(1)$, *and these are same as the equally denoted contours in* $L^{\vec{\varepsilon}}[k_1,k_2 \,|\, (k_1,k_2)]$ *(see Definition* (4.2)*).*

$$(\mathcal{S}L)^{\vec{\varepsilon}}[k_1,k_2,k_3 \,|\, (k_1,k_2)]\,(r,i;s,j) = \mathbf{1}_{\{k_1<r^*,\, s<k_3<k_2\}}\, c(r,i;s,j) \times$$
$$\frac{1}{w_c} \oint_{\gamma_{\tau_1}} d\zeta_1 \oint_{\gamma_{\tau_2}} d\zeta_2 \oint_{\gamma_{\tau_3}} d\zeta_3 \oint_{\vec{\gamma}_{R^{\vec{\varepsilon}}}} dz_{k_1+1} \cdots dz_{k_2} \left(\frac{1-\zeta_1}{1-z_1}\right)^{\mathbf{1}_{\{k_1=0\}}} \times$$
$$\frac{\prod_{k_1<k\leqslant k_2} G\big(z_k \,|\, \Delta_k(n,m,a)\big) \prod_{k_1<k<k_2} (z_k - z_{k+1})^{-1}\, (z_{k_1+1}-\zeta_1)^{-1} (z_{k_2}-\zeta_2)^{-1} (\zeta_2-\zeta_3)^{-1}}{G\big(\zeta_1 \,|\, i-n_{k_1}, \Delta_{k_1,r^*}(m,a)\big)\, G\big(\zeta_2 \,|\, n_{k_2}-n_{k_3-1}, \Delta_{k_3,k_2}(m,a)\big)\, G\big(\zeta_3 \,|\, n_{k_3-1}-j+1, \Delta_{s,k_3}(m,a)\big)}.$$

The difference here from $L^{\vec{\varepsilon}}[k_1,k_2,k_3 \,|\, (k_1,k_2)]$ *is that the number* n_{k_3} *is replaced by* n_{k_3-1} *in the second and third* G*-functions of the denominator.*

The matrix $L^{\vec{\varepsilon}}[k_1,k_2,k_3 \,|\, (k_1,k_2)]$ *is good and convergent. Its limit is* $(-1)^{k_2-k_1} F^{\vec{\varepsilon}}[k_1,k_2,k_3 \,|\, (k_1,k_2)]$. *The matrix* $(\mathcal{S}L)^{\vec{\varepsilon}}[k_1,k_2,k_3 \,|\, (k_1,k_2)]$ *is small.*

When $k_2 = p$ *there is an additional term in the representation above:*

$$L^{\vec{\varepsilon}}[k_1,p \,|\, (k_1,p)] \cdot B = \sum_{k_3=0}^{p} \big(1+\Theta(k_3 \,|\, s)\big)\Big[L^{\vec{\varepsilon}}[k_1,p,k_3 \,|\, (k_1,p)]\Big] - (1+\Theta(p \,|\, s)) \cdot L^{\vec{\varepsilon}}[k_1,p,p-1 \,|\, (k_1,p)]$$
$$- \sum_{k_3=0}^{p-1} (\mathcal{S}L)^{\vec{\varepsilon}}[k_1,k_2,k_3 \,|\, (k_1,k_2)]\Big].$$

Lemma 5.6. *Suppose* $0 \leqslant k_1 < k_2 < p$. *We have*

$$L^{\vec{\varepsilon}}[k_1 \,|\, (k_1,k_2)] \cdot B = \big(1+\Theta(k_2 \,|\, s)\big)\Big[L^{\vec{\varepsilon}}[k_1,k_2 \,|\, (k_1,k_2)] - (\mathcal{S}L)^{\vec{\varepsilon}}[k_1,k_2 \,|\, (k_1,k_2)]\Big], \quad \textit{where}$$

$$(\mathcal{S}L)^{\vec{\varepsilon}}[k_1,k_2 \,|\, (k_1,k_2)]\,(r,i;s,j) = \mathbf{1}_{\{k_1<r^*,\, s<k_2\}}\, c(r,i;s,j)\, \frac{1}{w_c} \oint_{\gamma_{\tau_1}} d\zeta_1 \oint_{\gamma_{\tau_2}} d\zeta_2 \oint_{\vec{\gamma}_{R^{\vec{\varepsilon}}}} dz_{k_1+1} \cdots dz_{k_2}$$
$$\frac{\prod_{k_1<k<k_2} G\big(z_k \,|\, \Delta_k(n,m,a)\big)\, G\big(z_{k_2} \,|\, 0, \Delta_{k_2}(m,a)\big) \prod_{k_1<k<k_2} (z_k - z_{k+1})^{-1} \left(\frac{1-\zeta_1}{1-z_1}\right)^{\mathbf{1}_{\{k_1=0\}}}}{G\big(\zeta_1 \,|\, i-n_{k_1}, \Delta_{k_1,r^*}(m,a)\big)\, G\big(\zeta_2 \,|\, n_{k_2-1}-j+1,\, \Delta_{s,k_2}(m,a)\big)\, (z_{k_1+1}-\zeta_1)\, (z_{k_2}-\zeta_2)}.$$

The contours are as in the lemma above. The matrix $(\mathcal{S}L)^{\vec{\varepsilon}}[k_1,k_2 \,|\, (k_1,k_2)]$ *is small.*

Lemma 5.7. *Suppose* $0 \leqslant k_1 \leqslant p$. *We have*

$$L[k_1,k_1 \,|\, \emptyset] \cdot B = \sum_{k_2=0}^{p} \big(1+\Theta(k_2 \,|\, s)\big)\Big[L[k_1,k_1,k_2 \,|\, \emptyset] - (\mathcal{S}L)[k_1,k_1,k_2 \,|\, \emptyset]\Big], \quad \textit{where}$$

$$L[k_1,k_1,k_2 \,|\, \emptyset]\,(r,i;s,j) = \mathbf{1}_{\{k_1<r^*,\, s<k_2<k_1\}}\, c(r,i;s,j)\, \frac{1}{w_c} \oint_{\gamma_{\tau_1}} d\zeta_1 \oint_{\gamma_{\tau_2}} d\zeta_2 \oint_{\gamma_{\tau_3}} d\zeta_3$$
$$\frac{(\zeta_1-\zeta_2)^{-1}(\zeta_2-\zeta_3)^{-1}}{G\big(\zeta_1 \,|\, i-n_{k_1}, \Delta_{k_1,r^*}(m,a)\big)\, G\big(\zeta_2 \,|\, \Delta_{k_2,k_1}(n,m,a)\big)\, G\big(\zeta_3 \,|\, n_{k_2}-j+1, \Delta_{s,k_2}(m,a)\big)}.$$

We arrange the radii $\tau_2 < \tau_1, \tau_3 < 1 - \sqrt{q}$.

$$(\mathcal{S}L)[k_1,k_1,k_2\,|\,\emptyset]\,(r,i;s,j) = \mathbf{1}_{\{k_1<r^*,\,s<k_2<k_1\}}\, c(r,i;s,j)\,\frac{1}{w_c}\oint_{\gamma_{\tau_1}} d\zeta_1 \oint_{\gamma_{\tau_2}} d\zeta_2 \oint_{\gamma_{\tau_3}} d\zeta_3$$
$$\frac{(\zeta_1-\zeta_2)^{-1}(\zeta_2-\zeta_3)^{-1}}{G\big(\zeta_1\,|\,i-n_{k_1},\Delta_{k_1,r^*}(m,a)\big)\,G\big(\zeta_2\,|\,n_{k_1}-n_{k_2-1},\Delta_{k_2,k_1}(m,a)\big)\,G\big(\zeta_3\,|\,n_{k_2-1}-j+1,\,\Delta_{s,k_2}(m,a)\big)}.$$

The difference from $L[k_1,k_1,k_2\,|\,\emptyset]$ is that the number n_{k_2} is replaced by n_{k_2-1} in the second and third G-functions of the denominator.

The matrix $L[k_1,k_1,k_2\,|\,\emptyset]$ is good and convergent with limit $F[k_1,k_1,k_2\,|\,\emptyset]$. The matrix $(\mathcal{S}L)[k_1,k_1,k_2\,|\,\emptyset]$ is small.

Lemma 5.8. *For the matrix $L[p\,|\,p]$ we have*

$$L[p\,|\,p]\cdot B(r,i;s,j) = \sum_{k=0}^{p}\big(1+\Theta(k\,|\,s)\big)L[p,k\,|\,p]\,(r,i;s,j) - \big(1+\Theta(p\,|\,s)\big)L[p,p-1\,|\,p]\,(r,i;s,j)$$
$$-\sum_{k=0}^{p}\big(1+\Theta(k\,|\,s)\big)(\mathcal{S}L)[p,k\,|\,p]\,(r,i;s,j).$$

The matrices $L[p,k\,|\,p]$ and $(\mathcal{S}L)[p,k\,|\,p]$ are as follows.

$$L[p,k\,|\,p]\,(r,i;s,j) = \mathbf{1}_{\{r=p,\,s<k<p\}}\, c(r,i;s,j)\,\frac{1}{w_c}\oint\limits_{\gamma_{\tau_2}} d\zeta_2 \oint\limits_{\gamma_{\tau_3}} d\zeta_3 \oint\limits_{\gamma_{R_p}(1)} dz_p$$
$$\frac{G\big(z_p\,|\,n_p-i,\Delta_p(m,a)\big)(z_p-\zeta_2)^{-1}(\zeta_2-\zeta_3)^{-1}}{G\big(\zeta_2\,|\,n_p-n_k,\Delta_{k,p}(m,a)\big)G\big(\zeta_3\,|\,n_k-j+1,\Delta_{s,k}(m,a)\big)},$$
$$(\mathcal{S}L)[p,k\,|\,p]\,(r,i;s,j) = \mathbf{1}_{\{r=p,\,s<k<p\}}\, c(r,i;s,j)\,\frac{1}{w_c}\oint\limits_{\gamma_{\tau_2}} d\zeta_2 \oint\limits_{\gamma_{\tau_3}} d\zeta_3 \oint\limits_{\gamma_{R_p}(1)} dz_p$$
$$\frac{G\big(z_p\,|\,n_p-i,\Delta_p(m,a)\big)(z_p-\zeta_2)^{-1}(\zeta_2-\zeta_3)^{-1}}{G\big(\zeta_2\,|\,n_p-n_{k-1},\Delta_{k,p}(m,a)\big)G\big(\zeta_3\,|\,n_{k-1}-j+1,\Delta_{s,k}(m,a)\big)}.$$

The radii are arranged such that $\tau_2 < \tau_3 < 1-\sqrt{q}$. (The difference between $L[p,k,\,|\,p]$ and $(\mathcal{S}L)[p,k,\,|\,p]$ is that the number n_k is changed to n_{k-1} in the second and third G-functions of the denominator.)

The matrix $L[p,k\,|\,p]$ is good and convergent with limit $-F[p,k\,|\,p]$. The matrix $(\mathcal{S}L)[p,k\,|\,p]$ is small.

Proposition 5.2. *The matrix $AB = (AB)_g + (AB)_s$, where $(AB)_g$ is good and convergent and $(AB)_s$ is small. This is due to the following reasons, which also provides the limit of $(AB)_g$. Recall from Definition*

(4.3) *that* $A = A_1 + A_2$. *Then* $(AB)_g = (A_1B)_g + (A_2B)_g$, *given as follows.*

$$(A_1B)_g(r,i;s,j) = \sum_{0\leqslant k_1,k_2\leqslant p} \Theta(r\,|\,k_1)\cdot(1+\Theta(k_2\,|\,s))\cdot L[k_1,k_1,k_2\,|\,\emptyset]\,(r,i;s,j).$$

$$(A_2B)_g(r,i;s,j) = \sum_{\substack{0\leqslant k_1,k_2,k_3\leqslant p,\,\vec{\varepsilon}\\ \text{satisfies (2.14)}}} (-1)^{\varepsilon[k_1,k_2]+k_1+k_2^*}\cdot\theta(r\,|\,\vec{\varepsilon})\times$$

$$\Big[(1+\Theta(k_3\,|\,s))L^{\vec{\varepsilon}}[k_1,k_2,k_3\,|\,(k_1,k_2)] - \mathbf{1}_{\{k_2=p,\,k_3=p-1\}}(1+\Theta(p\,|\,s))L^{\vec{\varepsilon}}[k_1,p,p-1\,|\,(k_1,p)] +$$

$$\mathbf{1}_{\{k_2<p,k_3=p\}}(1+\Theta(k_2\,|\,s))L^{\vec{\varepsilon}}[k_1,k_2\,|\,(k_1,k_2)] +$$

$$\mathbf{1}_{\{k_1=p-1,k_2=p\}}(1+\Theta(k_3\,|\,s))L[p,k_3\,|\,p] - \mathbf{1}_{\{k_1=p-1,k_2=p,k_3=p-1\}}(1+\Theta(p\,|\,s))L[p,p-1\,|\,p]\Big](c,i;s,j).$$

The summation variables k_i *range over* $0,1,\ldots,p$. *The matrix* $(AB)_s$ *looks the same as* $(AB)_g$ *except that every* L *is replaced by* $\mathcal{S}L$.

Proof. We see in Definition 4.3 that A is a weighted sum - involving the θ_ks - of the matrices $L[k,k\,|\,\emptyset]$, $L^{\vec{\varepsilon}}[k_1,k_2\,|\,(k_1,k_2)]$, $L^{\vec{\varepsilon}}[k_1\,|\,(k_1,k_2)]$ and $L[p\,|\,p]$. When we multiply A by B we replace every $L[\cdots]$ by $L[\cdots]\cdot B$. Then if we substitute the representation of these matrices by using Lemmas 5.5, 5.6, 5.7 and 5.8, we get the representation $(AB)_g + (AB)_s$ as given by the statement of the proposition. ∎

Lemma 5.4 along with Propositions 5.1 and 5.2 imply that the matrix $P = A - (AB)_g - B_1$ has limit F from (2.15). Specifically, the limit of B_1 is $F^{(0)}$. The limit of A_1 is $F^{(1)}$ and that of A_2 is $F^{(2)}$. The limit of $(A_1B)_g$ is $F^{(3)}$ and the one of $(A_2B)_g$ is $F^{(4)}$. Let us also remark that when comparing the matrix A with F, we see the factors $(-1)^{\varepsilon[k_1,k_2]+k_1+k_2^*}$ have become $(-1)^{\varepsilon[k_1,k_2]+\mathbf{1}_{\{k_2=p\}}}$. This is because limits of the $L^{\vec{\varepsilon}}$ are of the form $(-1)^{k_2-k_1}F^{\vec{\varepsilon}}$, and $k_2^* + k_2 = 2k_2 - \mathbf{1}_{\{k_2=p\}}$. Likewise for $L[p\,|\,p]$ with $k_1 = p-1$ and $k_2 = p$.

We then arrive at the conclusion of Theorem 1 once we have proved

Proposition 5.3. *The matrix* PB_2 *is small, where* $P = A - (AB)_g - B_1$ *and* B_2 *is from Lemma 5.4.*

The proof of this is in the next section. For the remainder of this section we will prove Proposition 5.1 and the aforementioned lemmas. The proofs will be on a case by case basis, where we consider each of the three types of matrices $L[k,k,\,|\,\emptyset]$, $L[k_1,k_2\,|\,(k_1,k_2)]$ and $L[k_1\,|\,(k_1,k_2)]$, and then prove the propositions claimed about them.

The following lemma will be used again and again to multiply matrices by B.

Lemma 5.9. *Suppose* $0 \leqslant N_1 < N_2$ *are integers and* $w_1 \neq w_2$ *belong to* $\mathbb{C}\setminus\{0,1,1-q\}$. *Then,*

$$\sum_{N_1<\ell\leqslant N_2} \frac{1}{G(w_1\,|\,n-\ell+1,m,a)\,G(w_2\,|\,\ell-n',m',a')} = \frac{w_c}{w_1-w_2}\times$$

$$\left[\frac{1}{G(w_1\,|\,n-N_2,m,a)\,G(w_2\,|\,N_2-n',m',a')} - \frac{1}{G(w_1\,|\,n-N_1,m,a)\,G(w_2\,|\,N_1-n',m',a')}\right]$$

Proof. Due to the group property of G, the sum over ℓ can be written as

$$\frac{1}{G(w_1\,|\,n,m,a)\,G(w_2\,|-n',m',a')}\sum_{N_1<\ell\leqslant N_2}\left(\frac{w_1}{w_c}\right)^{\ell-1}\left(\frac{w_c}{w_2}\right)^{\ell}.$$

The geometric sum evaluates to

$$\frac{w_c}{w_1-w_2}\left[(w_1/w_2)^{N_2}-(w_1/w_2)^{N_1}\right]=$$
$$\frac{w_c}{w_1-w_2}\left[\frac{1}{G(w_1\,|-N_2,0,0)\,G(w_2\,|\,N_2,0,0)}-\frac{1}{G(w_1\,|-N_1,0,0)\,G(w_2\,|\,N_1,0,0)}\right].$$

Then by the group property we obtain the expression on the r.h.s. of the identity . ■

Proof of Lemma 5.4. We have that

$$B^2(r,i;s,j)=\sum_{k=0}^{p}\sum_{n_{k-1}<\ell\leqslant n_k}B(r,i;k,\ell)B(k,\ell;s,j).$$

Let us recall

$$B(r,i;s,j)=\mathbf{1}_{\{s<r^*\}}c(r,i;s,j)\frac{1+\Theta(r\,|\,s)}{w_c}\oint_{\gamma_\tau}d\zeta\frac{1}{G\big(\zeta\,|\,i-j+1,\Delta_{s,r^*}(m,a)\big)}.$$

The conjugation factor satisfies $c(r,i;k,\ell)c(k,\ell;s,j)=c(r,i;s,j)$. Therefore,

$$B^2(r,i;s,j)=c(r,i;s,j)\sum_{k=0}^{p}\mathbf{1}_{\{k<r^*,\,s<k^*\}}\,(1+\Theta(r\,|\,k))\cdot(1+\Theta(k\,|\,s))\frac{1}{w_c^2}\oint_{\gamma_{\tau_1}}d\zeta_1\oint_{\gamma_{\tau_2}}d\zeta_2$$
$$\sum_{n_{k-1}<\ell\leqslant n_k}\frac{1}{G\big(\zeta_1\,|\,i-\ell+1,\Delta_{k,r^*}(m,a)\big)\,G\big(\zeta_2\,|\,\ell-j+1,\Delta_{s,k^*}(m,a)\big)}.$$

Observe that $k^*=k$ because $k<r^*<p$. By Lemma 5.9, the sum over ℓ gives the difference of the integrand of $L[k,k\,|\,\emptyset](r,i;s,j)$ from that of $(\mathcal{S}L)[k,k\,|\,\emptyset](r,i;s,j)$. Consequently, the expressions for B_1 and B_2 follow and we have $B^2=B_1-B_2$. That B_1 is good and convergent will follow due to every $L[k,k\,|\,\emptyset]$ being such, which will be shown in the proof of Proposition 5.2 below. ■

Throughout the remaining argument we will assume the following.

(1) The parameters t_k, x_k, ξ_k are bounded in absolute value by L and $\min_k\{\Delta_k t\}>0$.

(2) $C_{q,L}$ is a constant whose value may change from one appearance to the next, but depends on q and L only.

5.3.1. *Proof of claims regarding* $L^{\vec{\varepsilon}}[k_1,k_2\,|\,(k_1,k_2]]$. The matrix $L^{\vec{\varepsilon}}[k_1,k_2\,|\,(k_1,k_2]]$ has the from

$$L^{\vec{\varepsilon}}[k_1,k_2\,|\,(k_1,k_2]](r,i;s,j) = \mathbf{1}_{\{k_1<r^*,s^*<k_2\}} c(r,i;s,j)\frac{1}{w_c}\oint_{\gamma_{\tau_1}} d\zeta_1 \oint_{\gamma_{\tau_2}} d\zeta_2 \tag{5.16}$$

$$\frac{f(\zeta_1,\zeta_2)}{G(\zeta_1\,|\,i-n_{k_1},\Delta_{k_1,r^*}(m,a))\,G(\zeta_2\,|\,n_{k_2}-j+1,\Delta_{s^*,k_2}(m,a))}, \quad \text{where}$$

$$f(\zeta_1,\zeta_2) = \oint_{\vec{\gamma}_{R^{\vec{\varepsilon}}}} dz_{k_1+1}\cdots dz_{k_2}\,\frac{\prod_{k_1<k\leqslant k_2} G(z_k\,|\,\Delta_k(n,m,a))\,\left(\frac{1-\zeta_1}{1-z_1}\right)^{\mathbf{1}_{\{k_1=0\}}}}{\prod_{k_1<k<k_2}(z_k-z_{k+1})\,(z_{k_1+1}-\zeta_1)(z_{k_2}-\zeta_2)}.$$

Let us fix k_1,k_2 and $\vec{\varepsilon}$. Let F_T be the KPZ re-scaling of our matrix according to (5.4). The indices i and j on the (r,s)-block are re-scaled as

$$i = n_{r^*} + \lceil \nu_T u\rceil \quad \text{and} \quad j = n_{s^*} + \lceil \nu_T v\rceil\,. \tag{5.17}$$

It is convenient to ignore the rounding as it makes no difference in the asymptotic analysis. Consequently,

$$\begin{aligned} i-n_{k_1} &= \Delta_{k_1,r^*}t\,T - c_1(\Delta_{k_1,r^*}x)\cdot(\Delta_{k_1,r^*}t\,T)^{\frac{2}{3}} + c_0\frac{u}{(\Delta_{k_1,r^*}t)^{1/3}}(\Delta_{k_1,r^*}t\,T)^{\frac{1}{3}}\,,\\ \Delta_{k_1,r^*}m &= \Delta_{k_1,r^*}tT + c_1(\Delta_{k_1,r^*}x)\cdot(\Delta_{k_1,r^*}t\,T)^{\frac{2}{3}}\,,\\ \Delta_{k_1,r^*}a &= c_2\,\Delta_{k_1,r^*}t\,T + c_3(\Delta_{k_1,r^*}\xi)\cdot(\Delta_{k_1,r^*}t\,T)^{\frac{1}{3}}. \end{aligned}\tag{5.18}$$

Similarly,

$$\begin{aligned} n_{k_2}-j &= \Delta_{s^*,k_2}t\,T - c_1(\Delta_{s^*,k_2}x)\cdot(\Delta_{s^*,k_2}t\,T)^{\frac{2}{3}} + c_0\frac{-v}{(\Delta_{s^*,k_2}t)^{1/3}}(\Delta_{s^*,k_2}t\,T)^{\frac{1}{3}}\,,\\ \Delta_{k_1,r^*}m &= \Delta_{s^*,k_2}t\,T + c_1(\Delta_{s^*,k_2}x)\cdot(\Delta_{s^*,k_2}t\,T)^{\frac{2}{3}}\,,\\ \Delta_{k_1,r^*}a &= c_2\,\Delta_{s^*,k_2}t\,T + c_3(\Delta_{s^*,k_2}\xi)\cdot(\Delta_{s^*,k_2}t\,T)^{\frac{1}{3}}. \end{aligned}\tag{5.19}$$

We note that $\Delta_{k_1,r^*}t>0$ and $\Delta_{s^*,k_2}t>0$ due to the conditions $k_1<r^*$ and $s^*<k_2$.

Recalling Definition 5.2, choose the contours γ_{τ_1} and γ_{τ_2} as follows.

$$\gamma_{\tau_1} = w_0(\sigma_1,d_1)\ \text{ with }\ K := \Delta_{k_1,r^*}t\,T;\ \ \gamma_{\tau_2} = w_0(\sigma_2,d_2)\ \text{ with }\ K := \Delta_{s^*,k_2}t\,T.$$

The choices for d_1 and d_2 will be made later.

With the re-scaling (5.17) the conjugation factor satisfies

$$c(r,i;s,j) = e^{\mu(v-u)}\,(1+C_{q,L}T^{-1/3}). \tag{5.20}$$

Proof that $L^{\vec{\varepsilon}}[k_1,k_2\,|\,(k_1,k_2]]$ **is good**. From Lemma 5.3 we see there is a choice of $d_1=d(u)$ such that we have the following uniformly in $\zeta_1=\zeta_1(\sigma_1)\in w_0(\sigma_1,d_1)$.

$$|G(\zeta_1(\sigma_1)\,|\,i-n_{k_1},\Delta_{k_1,r^*}(m,a))|^{-1} \leqslant C_3 e^{-C_4\sigma_1^2+\Psi\left(u/(\Delta_{k_1,r^*}t)^{1/3}\right)}.$$

Recall $\Psi(x) = -\mu_1 \cdot (x)_-^{3/2} + \mu_2 \cdot (x)_+$. Also, there is a choice of $d_2 = d(-\nu)$ such that the following holds uniformly in $\zeta_2 = \zeta_2(\sigma_2) \in w_0(\sigma_2, d_2)$.

$$|G(\zeta_2(\sigma_2)\,|\,n_{k_2} - j + 1, \Delta_{s^*,k_2}(m,a))|^{-1} \leqslant C_3 e^{-C_4\sigma_2^2 + \Psi\left(-\nu/(\Delta_{s^*,k_2} t)^{1/3}\right)}.$$

We will see below that f from (5.16) satisfies the following uniformly in σ_1 and σ_2.

$$|f(\zeta_1(\sigma_1), \zeta_2(\sigma_2))| \leqslant C_{q,L} T^{1/3}. \tag{5.21}$$

When we change variables $\zeta_1 \mapsto \sigma_1$ and $\zeta_2 \mapsto \sigma_2$ we have $|d\zeta_\ell / d\sigma_\ell| \leqslant C_{q,L} T^{-1/3}$ for $\ell = 1, 2$. The conjugation factor also satisfies (5.20). Therefore,

$$\begin{aligned}
|F_T(r, u; s, \nu)| &\leqslant C_{q,L}\, \nu_T\, T^{-2/3}\, e^{\mu(\nu - u)} \int_{\mathbb{R}^2} d\sigma_2 d\sigma_2\, |f(\zeta_1(\sigma_1), \zeta_2(\sigma_2))|\, e^{-C_4(\sigma_1^2+\sigma_2^2)} \times \\
&\quad \times e^{\Psi\left((u/(\Delta_{k_1,r^*} t)^{1/3}\right)} \cdot e^{\Psi\left(-\nu/(\Delta_{s^*,k_2} t)^{1/3}\right)} \\
&\leqslant C_{q,L} e^{-\mu u + \Psi\left((u/(\Delta_{k_1,r^*} t)^{1/3}\right)} \cdot e^{\mu\nu + \Psi\left(-\nu/(\Delta_{s^*,k_2} t)^{1/3}\right)}.
\end{aligned}$$

Recall from (5.15) that $e^{-\mu x + \Psi(x/\Delta)}$ is bounded and integrable over $\mathbb{R}$ if μ satisfies the bound from (5.13) and $\Delta \geqslant \min_k \{(\Delta_k t)^{1/3}\}$. This is the case for us and the matrix is good.

Proof of estimate (5.21) **for** $f(\zeta_1, \zeta_2)$. First, $|(1-\zeta_1)/(1-z_1)| \leqslant 2/(1-q)$. Suppose that $\zeta_1 \in w_0(\sigma, d_1)$ for some d_1 and $K = \kappa_1 T$, and $z_{k_1+1} \in w_1(\sigma, d_2)$ for some d_2 and $K = \kappa_2 T$. Then $|\zeta_1 - z_{k_1+1}| \geqslant T^{-1/3}((d_1/\kappa_1) + (d_2/\kappa_2))$. In our case, $d_1, d_2, \kappa_2, \kappa_2$ all remain uniformly positive in T, and depends on q and L. Therefore, $|\zeta_1 - z_{k_1+1}|^{-1} \leqslant C_{q,L} T^{1/3}$. Similarly, $|\zeta_2 - z_{k_2}|^{-1} \leqslant C_{q,L} T^{1/3}$ if $z_{k_2} \in w_1(\sigma, d_2)$.

The parameters $\Delta_k(n, m, a)$ are re-scaled according to

$$\begin{aligned}
\Delta_k n &= \Delta_k t, T - c_1(\Delta_k x) \cdot (\Delta_k t\, T)^{\frac{2}{3}}, \\
\Delta_k m &= \Delta_k t, T + c_1(\Delta_k x) \cdot (\Delta_k t\, T)^{\frac{2}{3}}, \\
\Delta_k a &= c_2\, \Delta_k t, T + c_3(\Delta_k \xi) \cdot (\Delta_k t\, T)^{\frac{1}{3}}.
\end{aligned} \tag{5.22}$$

We choose z_k to lie on the contour $w_1(\sigma_k, D_k)$ with the choice $K = \Delta_k t\, T$. The number D_k is chosen so that the estimate (5.12) from Lemma 5.3 holds, namely, uniformly in σ_k,

$$|G(z_k(\sigma_k)\,|\,\Delta_k(n, m, a)| \leqslant C_3 e^{-C_4 \sigma_k^2}.$$

This is for every $k_1 < k \leqslant k_2$.

We need the D_ks to be ordered according to $\vec{\varepsilon}$. The D_ks may be chosen from an interval with length of order $T^{1/3}$. So we can choose them from the interval $[1, 2p]$, say, which ensures that they can be ordered accordingly and also that their pairwise distance is at least 1. Consequently, $|z_k - z_{k+1}|^{-1} \leqslant C_{q,L} T^{1/3}$ for every k.

When we change variables $z_k \mapsto \sigma_k$ we have $|dz_k/d\sigma_k| \leqslant C_{q,L}T^{-1/3}$. Thus, if $\zeta_1 \in w_0(\sigma, d_1)$ and $\zeta_2 \in w_0(\sigma', d_2)$, then uniformly in ζ_1 and ζ_2,

$$\begin{aligned}|f(\zeta_1,\zeta_2)| &\leqslant C_{q,L}\,(T^{-1/3})^{k_2-k_1}\int_{\mathbb{R}^{k_2-k_1}} d\sigma_{k_1+1}\cdots d\sigma_{k_2} e^{-C_4\sum_k \sigma_k^2}\cdot (T^{1/3})^{k_2-k_1-1+2}\\ &\leqslant C_{q,L}T^{1/3}.\end{aligned}$$

Proof that $L^{\vec{\varepsilon}}[k_1,k_2\,|\,(k_1,k_2]]$ is convergent. Now we assume the kernel variables u and v in F_T remain bounded and we are on the (r,s)-block. We will choose contours for all the variables in the following way.

$$\begin{aligned}\zeta_1 &= \zeta_1(\hat\sigma_1) \in w_0\left(\frac{c_4}{\sqrt{q}}\hat\sigma_1, \frac{c_4 d_1}{\sqrt{q}}\right), \quad K := \Delta_{k_1,r^*}t\,T.\\ \zeta_2 &= \zeta_1(\hat\sigma_2) \in w_0\left(\frac{c_4}{\sqrt{q}}\hat\sigma_2, \frac{c_4 d_2}{\sqrt{q}}\right), \quad K := \Delta_{s^*,k_2}t\,T.\\ z_k &= z_\ell(\sigma_k) \in w_1\left(\frac{c_4}{\sqrt{q}}\sigma_k, \frac{c_4 D_k}{\sqrt{q}}\right), \quad K := \Delta_k t\,T.\end{aligned}$$

The constant c_4 is from (5.8). The numbers d_1 and d_2 are as in the proof of goodness so that the estimate (5.11) holds. Since u and v are bounded, we may absorb the terms $e^{\Psi(u)}$ and $e^{\Psi(-v)}$ into the constant C_3 of the estimate. The number D_k are chosen so that the estimate (5.12) holds. They are also to be ordered according to $\vec{\varepsilon}$. As before, we may choose them so that they have pairwise distance at least 1 and are ordered accordingly; the condition of the ordering is (2.13).

Due to this choice of contours, arguing as before, we find the following estimates. We have $z_k = z_k(\sigma_k)$ and $\zeta_\ell = \zeta_\ell(\hat\sigma_\ell)$.

$$\frac{\prod_k |G(z_k\,|\,\Delta_k(n,m,a))|}{|G(\zeta_1\,|\,i-n_{k_1},\Delta_{k_1,r^*}(m,a))\cdot G(\zeta_2\,|\,n_{k_2}-j+1,\Delta_{s^*,k_2}(m,a))|} \leqslant C_{q,L}\,e^{-C_4(\sum_k \sigma_k^2+\hat\sigma_1^2+\hat\sigma_2^2)}.$$

$$\nu_T\cdot\Big|\prod_{k_1<k<k_2}(z_k-z_{k+1})^{-1}(z_{k_1+1}-\zeta_1)^{-1}(z_{k_2}-\zeta_2)^{-1}\Big|\cdot\prod_{k_1<k\leqslant k_2}\Big|\frac{dz_k}{d\sigma_k}\Big|\cdot\prod_{\ell=1,2}\Big|\frac{d\hat\zeta_\ell}{\hat\sigma_\ell}\Big| \leqslant C_{q,L}.$$

These estimates allow us to use the dominated convergence theorem to get the limit of the integral in $F_T(r,u;s,v)$. So we consider the point-wise limit of the integral.

Suppose σ_k and $\hat\sigma_\ell$ lie on compact subsets of $\mathbb{R}$. We have

$$\begin{aligned}\zeta_1(\hat\sigma_1) &= w_c + \frac{c_4}{(\Delta_{k_1,r^*}t\,T)^{1/3}}(\mathbf{i}\hat\sigma_1+d_1) + C_{q,L}T^{-2/3}.\\ \zeta_2(\hat\sigma_2) &= w_c + \frac{c_4}{(\Delta_{s^*,k_2}t\,T)^{1/3}}(\mathbf{i}\hat\sigma_2+d_2) + C_{q,L}T^{-2/3}.\\ z_k(\sigma_k) &= w_c + \frac{c_4}{(\Delta_k t\,T)^{1/3}}(-\mathbf{i}\sigma_k+D_k) + C_{q,L}T^{-2/3}.\end{aligned}$$

Let us write $z_k' = (-\mathbf{i}\sigma_k + D_k)/\Delta_k t$, $\zeta_1' = (\mathbf{i}\hat\sigma_1+d_1)/(\Delta_{k_1,r^*}t)$ and $\zeta_2' = (\mathbf{i}\hat\sigma_2+d_2)/(\Delta_{s^*,k_2}t)$. With the new variables, in the large T limit, the contour γ_{τ_ℓ} becomes the vertical contour Γ_{-d_ℓ} intersecting the real axis at $-d_\ell$ (recall ζ_ℓ' now remains bounded). The contour $\gamma_{R_k}(1)$ becomes the vertical contour Γ_{D_k} oriented downward. It is downward because $\gamma_{R_k}(1)$ crosses the real axis at the point $1-R_k$ (which is the one near w_c) in the downward direction. If we re-orient the contours upward then we gain a factor of $(-1)^{k_2-k_1}$.

We see from Lemma 5.2 that

$$\begin{aligned}
G\big(z_k \,|\, \Delta_k(n,m,a)\big) &\longrightarrow \mathcal{G}\big(\Delta_k t\cdot z_k' \,|\, 1, \Delta_k(x,\xi)\big) = \mathcal{G}\big(z_k' \,|\, \Delta_k(t,x,\xi)\big). \\
G\big(\zeta_1 \,|\, i-n_{k_1}, \Delta_{k_1,r^*}(m,a)\big) &\longrightarrow \mathcal{G}\big(\Delta_{k_1,r^*} t\cdot \zeta_1' \,|\, 1, \Delta_{k_1,r^*}x,\, \Delta_{k_1,r^*}\xi - (\Delta_{k_1,r^*}t)^{-1/3}u\big) \\
&= \mathcal{G}\big(\zeta_1' \,|\, \Delta_{k_1,r^*}(t,x,\xi)\big)\, e^{\zeta_1' u}. \\
G\big(\zeta_2 \,|\, n_{k_2}-j+1, \Delta_{s^*,k_2}(m,a)\big) &\longrightarrow \mathcal{G}\big(\Delta_{s^*,k_2} t\cdot \zeta_2' \,|\, 1, \Delta_{s^*,k_2}x,\, \Delta_{s^*,k_2}\xi + (\Delta_{s^*,k_2}t)^{-1/3}v\big) \\
&= \mathcal{G}\big(\zeta_1' \,|\, \Delta_{s^*,k_2}(t,x,\xi)\big)\, e^{-\zeta_2' v}.
\end{aligned}$$

These limits are uniformly so in ζ_ℓ and z_k, as well as in u and v, because these variables are now restricted to compact subsets of their domains. We also have the following.

a) $\displaystyle\prod_{k_1<k<k_2} (z_k - z_{k+1}) = (c_4)^{k_2-k_1-1}(T^{-1/3})^{k_2-k_1-1} \prod_{k_1<k<k_2} (z_k' - z_{k+1}') + C_{q,L}(T^{-1/3})^{k_2-k_1}.$

b) $\displaystyle\prod_{k_1<k\leqslant k_2} dz_k\cdot \nu_T = c_0(c_4)^{k_2-k_1}(T^{-1/3})^{k_2-k_1-1} \prod_{k_1<k\leqslant k_2} dz_k' + C_{q,L}(T^{-1/3})^{k_2-k_1}.$

c) $(z_k-\zeta_\ell)^{-1}d\zeta_\ell = (z_k'-\zeta_\ell')^{-1}d\zeta_\ell' + C_{q,L}T^{-1/3}$; $(k,\ell) = (k_1+1,1)$ or $(k_2,2)$.

Next, we have that $c_0c_4 = 1-\sqrt{q} = w_c$, which is a factor we obtain from the ratio of the second product above to the first's. This cancels the factor $1/w_c$ in $F_T(r,u;s,v)$. Also, as $T\to\infty$, the term $(\frac{1-\zeta_1}{1-z_1})^{\mathbf{1}_{\{k_1=0\}}} \to 1$ and the conjugation factor $c(r,i;s,j)\to c(r,u;s,v) = e^{\mu(v-u)}$ by (5.20).

Putting all this together we see that the limit of the kernel $F_T(r,u;s,v)$ is the kernel $(-1)^{k_2-k_1}\times F^{\vec{\varepsilon}}[k_1,k_2\,|\,(k_1,k_2]](r,u;s,v)$, the latter from part (3) of Definition 2.1. This proves part (1) of Proposition 5.1. This same argument will be used with minor changes to show goodness and convergence of all the other matrices. ■

Proof of Lemma 5.5. First we will prove the decomposition of $L^{\vec{\varepsilon}}[k_1,k_2\,|\,(k_1,k_2]]\cdot B$ given in the lemma. We keep to the notation there. Using Lemma 5.9 we find that

$$L^{\vec{\varepsilon}}[k_1,k_2\,|\,(k_1,k_2]]\cdot B = \sum_{k_3=0}^{p} \big(1+\Theta(k_3\,|\,s)\big)\big[\hat{L}_{k_3} - \mathcal{S}\hat{L}_{k_3}\big].$$

$$\begin{aligned}
&\hat{L}_{k_3}(r,i;s,j) = \mathbf{1}_{\{k_1<r^*,\, s<k_3^*<k_2\}}\, c(r,i;s,j)\, \times \\
&\quad \frac{1}{w_c}\oint_{\gamma_{\tau_1}} d\zeta_1 \oint_{\gamma_{\tau_2}} d\zeta_2 \oint_{\gamma_{\tau_3}} d\zeta_3 \oint_{\vec{\gamma}_{R^{\vec{\varepsilon}}}} dz_{k_1+1}\cdots dz_{k_2} \left(\frac{1-\zeta_1}{1-z_1}\right)^{\mathbf{1}_{\{k_1=0\}}} \times \\
&\frac{\prod_{k_1<k\leqslant k_2} G\big(z_k\,|\,\Delta_k(n,m,a)\big) \prod_{k_1<k<k_2}(z_k-z_{k+1})^{-1}\,(z_{k_1+1}-\zeta_1)^{-1}(z_{k_2}-\zeta_2)^{-1}(\zeta_2-\zeta_3)^{-1}}{G\big(\zeta_1\,|\,i-n_{k_1},\Delta_{k_1,r^*}(m,a)\big)\, G\big(\zeta_2\,|\,n_{k_2}-n_{k_3},\Delta_{k_3^*,k_2}(m,a)\big)\, G\big(\zeta_3\,|\,n_{k_3}-j+1,\Delta_{s,k_3^*}(m,a)\big)}.
\end{aligned}$$

The matrix $\mathcal{S}\hat{L}_{k_3}$ looks the same as $\hat{L}_{k_3}$ with the difference being that n_{k_3} is changed to n_{k_3-1} in the two G-functions corresponding to variables ζ_2 and ζ_3.

The matrix $\hat{L}_{k_3}$ looks the same as $L^{\vec{\varepsilon}}[k_1,k_2,k_3 \,|\, (k_1,k_2]]$ except that k_3^* appears instead of k_3 in $\mathbf{1}_{\{s<k_3^*<k_2\}}$, $\Delta_{k_3^*,k_2}(m,a)$ and $\Delta_{s,k_3^*}(m,a)$. Now $k_3^* = k_3$ if $k_3 < p$. An exception occurs if $k_3 = k_2 = p$. In this case $n_{k_2} - n_{k_3} = 0$, so there is no pole at $\zeta_2 = 0$ in the integrand. The ζ_2-contour is the innermost one since $\tau_2 < \tau_3$, and it can be contracted to zero. So we may assume $k_3 < p$, and then replace k_3^* with k_3 in the above. This results in $L^{\vec{\varepsilon}}[k_1,k_2,k_3 \,|\, (k_1,k_2]]$.

Now consider $\mathcal{S}\hat{L}_{k_3}$. It will also equal $(\mathcal{S}L)^{\vec{\varepsilon}}[k_1,k_2,k_3 \,|\, (k_1,k_2]]$ unless $k_3 = k_2 = p$. In the latter case, since $k_3^* = p-1$, the matrix is $L^{\vec{\varepsilon}}[k_1,p,p-1 \,|\, (k_1,k_2]]$. Accounting for this case we get the representation of $L^{\vec{\varepsilon}}[k_1,k_2 \,|\, (k_1,k_2]] \cdot B$ given by the lemma.

Next we prove that $L^{\vec{\varepsilon}}[k_1,k_2,k_3 \,|\, (k_1,k_2]]$, which we simply write L, is good. Fix k_1,k_2,k_3 and an (r,s)-block such that $k_1 < r^*$ and $s < k_3 < k_2$. The argument is the same as the one for goodness of $L^{\vec{\varepsilon}}[k_1,k_2 \,|\, (k_1,k_2]]$ since these matrices have the same structure. The variable ζ_3 now has the same role as the variable ζ_2 did for $L^{\vec{\varepsilon}}[k_1,k_2 \,|\, (k_1,k_2]]$, i.e., it carries the j-index. The difference now is that ζ_2 appears in $(\zeta_2-\zeta_3)^{-1}/G(\zeta_2 \,|\, \Delta_{k_3,k_2}(n,m,a))$.

We choose ζ_2 to lie on the contour $\gamma_{\tau_2} = w_0(\sigma_2, d_2)$ with $K \coloneqq \Delta_{k_3,k_2} t\, T$. The number d_2 is to be chosen so that we have the estimate (5.11) from Lemma 5.3, i.e.,

$$\left| G\big(\zeta_2(\sigma_2) \,|\, \Delta_{k_3,k_2}(n,m,a)\big)\right|^{-1} \leqslant C_3 e^{-C_4 \sigma_2^2}.$$

As before, ζ_3 is chosen to lie on $\gamma_{\tau_3} = w_0(\sigma_3, d(-\nu))$ so that we have the estimate

$$\left| G\big(\zeta_3(\sigma_3) \,|\, \Delta_{s,k_3} n - \nu_T \nu, \Delta_{s,k_3}(m,a)\big)\right|^{-1} \leqslant C_3 e^{-C_4\sigma_3^2 + \Psi\left(-\nu/(\Delta_{s,k_3} t)^{1/3}\right)}.$$

We have $|\zeta_2-\zeta_3|^{-1} \leqslant C_{q,L} T^{1/3}$ uniformly over the contours, and also $|d\zeta_2/d\sigma_2| \leqslant C_{q,L} T^{-1/3}$.

Due to the term $(\zeta_2-\zeta_3)^{-1}$ we have to ensure that the contours are chosen so that they remain ordered, i.e., $\tau_2 < \tau_3$. This means we want $d(-\nu) < (d_2-1)\cdot(\Delta_{s,k_3} t/\Delta_{k_3,k_2} t)^{1/3} \leqslant C_{q,L} d_2$, say. Since the column block $s < p$, we have $\nu \leqslant 0$, and both d_2 and $d(-\nu)$ can be chosen from intervals of order $T^{1/3}$ in length. So we can order the contours.

Using the estimates above and arguing as in the proof of goodness of $L^{\vec{\varepsilon}}[k_1,k_2 \,|\, (k_1,k_2]]$ we find that L is good as well. Specifically, if F_T is the re-scaled kernel of L according to (5.4), then

$$|F_T(r,u;s,\nu)| \leqslant C_{q,L} e^{-\mu u + \Psi\left(u/(\Delta_{k_1,r^*} t)^{1/3}\right)} \cdot e^{\mu\nu + \Psi\left(-\nu/(\Delta_{s,k_3} t)^{1/3}\right)}.$$

This bound certifies goodness.

Now we argue that L is convergent to $(-1)^{k_2-k_1} F^{\vec{\varepsilon}}[k_1,k_2,k_3 \,|\, (k_1,k_2]]$. This is the same as the earlier proof of convergence of $L^{\vec{\varepsilon}}[k_1,k_2 \,|\, (k_1,k_2]]$. In the KPZ scaling limit the function $G(\zeta_2 \,|\, \Delta_{k_3,k_2}(n,m,a))$ converges to $\mathcal{G}(\zeta_2' \,|\, \Delta_k(t,x,\xi))$. Then the KPZ re-scaled kernel is seen to converge as before.

Finally, we prove that the matrices $\mathcal{S}L^{\vec{\varepsilon}}[k_1,k_2,k_3\,|\,(k_1,k_2)]$ are small. Let us fix k_1,k_2,k_3, and consider a block (r,s) such that $k_1<r^*$ and $s<k_3<k_2$. We have that

$$\mathcal{S}L^{\vec{\varepsilon}}[k_1,k_2,k_3\,|\,(k_1,k_2)]\,(r,i;s,j)=\frac{c(r,i;s,j)}{w_c}\oint_{\gamma_{\tau_1}}d\zeta_1\oint_{\gamma_{\tau_2}}d\zeta_2\oint_{\gamma_{\tau_3}}d\zeta_3\,f(\zeta_1,\zeta_2)\times$$
$$\frac{(\zeta_2-\zeta_3)^{-1}}{G\big(\zeta_1\,|\,i-n_{k_1},\Delta_{k_1,r^*}(m,a)\big)G\big(\zeta_2\,|\,n_{k_2}-n_{k_3-1},\Delta_{k_3,k_2}(m,a)\big)G\big(\zeta_3\,|\,n_{k_3-1}-j+1,\Delta_{s,k_3}(m,a)\big)}.$$

The function $f(\zeta_1,\zeta_2)$ is from (5.16) and satisfies the bound (5.21). The contours are ordered such that $\tau_2<\tau_3$.

For convenience, introduce

$$\Delta_1=(\Delta_{k_1,r^*}t)^{1/3},\ \Delta_2=(\Delta_{k_3,k_2}t)^{1/3},\ \Delta_3=(\Delta_{s,k_3}t)^{1/3},\ \lambda=\frac{\Delta_{k_3}n}{\nu_T}=\frac{\Delta_{k_3}t}{c_0}T^{2/3}+C_{q,L}T^{1/3}.$$

We find, ignoring rounding, that

$$\begin{aligned}(i-n_{k_1},\Delta_{k_1,r^*}m,\Delta_{k_1,r^*}a)&=\Delta_{k_1,r^*}(n,m,a)+c_0\big(u/\Delta_1,0,0\big)\cdot(\Delta_{k_1,r^*}t\,T)^{1/3},\\(n_{k_2}-n_{k_3-1},\Delta_{k_3,k_2}m,\Delta_{k_3,k_2}a)&=\Delta_{k_3,k_2}(n,m,a)+c_0\big(\lambda/\Delta_2,0,0\big)\cdot(\Delta_{k_3,k_2}t\,T)^{1/3},\\(n_{k_3-1}-j,\Delta_{s,k_3}m,\Delta_{s,k_3}a)&=\Delta_{s,k_3}(n,m,a)+c_0\big(-(\nu+\lambda)/\Delta_3,0,0\big)\cdot(\Delta_{s,k_3}t\,T)^{1/3}.\end{aligned}$$

Note $n_{k_3-1}-j\geqslant 0$ because $j\in(n_{s-1},n_s]$ and $s<k_3$.

Now we choose contours for the variables. We choose γ_{τ_1} to be $w_0(\sigma_1,d(u))$ with $K:=(\Delta_1)^3T$ such that we have the estimate (5.11) , namely,

$$|G(\zeta_1(\sigma_1)\,|\,i-n_{k_1},\Delta_{k_1,r^*}(m,a))|^{-1}\leqslant C_3e^{-C_4\sigma_1^2+\Psi(u/\Delta_1)}.$$

Next we choose γ_{τ_2} to be $w_0(\sigma_2,d(\lambda))$ with $K:=(\Delta_2)^3T$ such that we have

$$|G(\zeta_2(\sigma_2)\,|\,n_{k_2}-n_{k_3-1},\Delta_{k_3,k_2}(m,a))|^{-1}\leqslant C_3e^{-C_4\sigma_2^2+\Psi(\lambda/\Delta_2)}.$$

Finally, γ_{τ_3} is chosen to be $w_0(\sigma_3,d(-\nu-\lambda))$ with $K:=(\Delta_3)^3T$ such that

$$|G(\zeta_3(\sigma_3)\,|\,n_{k_3-1}-j+1,\Delta_{s,k_3}(m,a)|^{-1}\leqslant C_3e^{-C_4\sigma_3^2+\Psi\big(-(\nu+\lambda)/\Delta_3\big)}.$$

We have to maintain the ordering $\tau_2<\tau_3$ due to the term $(\zeta_2-\zeta_3)^{-1}$ in the integrand. So we should have $d(-\nu-\lambda)/\Delta_3<(d(\lambda)-1)/\Delta_2$, say. We know that $d(\lambda)/\Delta_2$ may belong to the interval $[C_1/\Delta_2,C_2T^{1/3}]$ if T is sufficiently large in terms of q and L. If $\nu+\lambda\leqslant 0$ then $d(-(\nu+\lambda))/\Delta_3$ may belong to $[C_1/\Delta_3,C_2T^{1/3}]$, and we may order the contours as we wish.

On the other hand, if $\nu+\lambda>0$ then $d(-\nu-\lambda)/\Delta_3$ may belong to the interval

$$[C_1/\Delta_3+\delta(\nu+\lambda)^{1/2}/\Delta_3^{3/2},C_2T^{1/3}].$$

Since $d(\lambda)/\Delta_2$ belongs to $[C_1/\Delta_2,C_2T^{1/3}]$, we can ensure that $d(-\nu-\lambda)/\Delta_3<(d(\lambda)-1)/\Delta_2$ for all sufficiently large T so long as $\delta(\nu+\lambda)^{1/2}<C_2\Delta_3^{3/2}T^{1/3}-C_1\Delta_3^{1/2}$. Now observe that $\nu\leqslant 0$ because index j belongs to column block s with $s<p$ due to $s<k_3<k_2$. Therefore, $(\nu+\lambda)^{1/2}\leqslant\lambda^{1/2}=(\Delta_{k_3}t/c_0)^{1/2}\,T^{1/3}+C_{q,L}T^{1/6}$. So we are fine if $\delta<C_2c_0^{1/2}\Delta_3^{3/2}(\Delta_{k_3}t)^{-1/2}$. We note that

$\Delta_3^{3/2} \geqslant \min_k\{(\Delta_k t)^{1/2}\}$ and $\Delta_{k_3} t \leqslant t_p$. So δ satisfies the required bound as it is chosen according to (5.13).

Let $F_T(r,u;s,v)$ be our matrix re-scaled according to (5.4). Recall $|f(\zeta_1,\zeta_2)| \leqslant C_{q,L} T^{1/3}$ according to (5.21). Then, using the above bounds for the G-functions and arguing as in the proof of goodness of $L^{\vec{\varepsilon}}[k_1,k_2 \,|\, (k_1,k_2)]$, we find that

$$\begin{aligned}|F_T(r,u;s,v)| &\leqslant C_{q,L}\, e^{\mu(v-u)} e^{\Psi(u/\Delta_1)+\Psi(\lambda/\Delta_2)+\Psi(-(v+\lambda)/\Delta_3)} \\ &= \underbrace{C_{q,L}\, e^{-\mu\lambda+\Psi(\lambda/\Delta_2)}}_{\eta_T} \cdot e^{-\mu u+\Psi(u/\Delta_1)} \cdot e^{\mu(v+\lambda)+\Psi(-(v+\lambda)/\Delta_3)}.\end{aligned}$$

Note that every $\Delta_\ell \geqslant \min_k\{(\Delta_k t)^{1/3}\}$ and μ satisfies (5.13). Therefore, from (5.15), we have that $\eta_T \to 0$ as $T \to \infty$ due to $\lambda \to \infty$. We also see that the functions $e^{-\mu u+\Psi(u/\Delta_1)}$ and $e^{\mu(v+\lambda)+\Psi(-(v+\lambda)/\Delta_3)}$ are bounded and integrable over the reals. This certifies smallness of $\mathcal{S}L^{\vec{\varepsilon}}[k_1,k_2,k_3 \,|\, (k_1,k_2)]$. ■

5.3.2. *Proof of claims regarding* $L^{\vec{\varepsilon}}[k_1 \,|\, (k_1,k_2)]$. We will first prove $L^{\vec{\varepsilon}}[k_1 \,|\, (k_1,k_2)]$ is good and convergent as stated by Proposition 5.1. Then we will prove Lemma 5.6.

Proof that it is good and convergent. Fix $\vec{\varepsilon}$ and $k_1 < k_2$. Note $L^{\vec{\varepsilon}}[k_1 \,|\, (k_1,k_2)]$ has non-zero blocks only of column block $s = k_2 < p$. Consider the block (r,s) such that $k_1 < r^*$ and $s = k_2 < p$. On this block the matrix has form

(5.23)

$$\begin{aligned}L^{\vec{\varepsilon}}[k_1 \,|\, (k_1,k_2)](r,i;s,j) &= c(r,i;s,j)\frac{1}{w_c}\oint\limits_{\gamma_{\tau_1}} d\zeta_1 \oint\limits_{\gamma R_{k_2}(1)} dz_{k_2}\, f(\zeta_1,z_{k_2}) \times \\ &\times \frac{G\big(z_{k_2} \,|\, j-n_{k_2-1}-1, \Delta_{k_2}(m,a)\big)}{G\big(\zeta_1 \,|\, i-n_{k_1}, \Delta_{k_1,r^*}(m,a)\big)}.\end{aligned}$$

$$f(\zeta_1,z_{k_2}) = \oint\limits_{\gamma_{R_{k_1+1}}(1)} dz_{k_1+1} \cdots \oint\limits_{\gamma_{R_{k_2-1}}(1)} dz_{k_2-1}\, \frac{\prod_{k_1<k<k_2} G(z_k \,|\, \Delta_k(n,m,a)) \left(\frac{1-\zeta_1}{1-z_1}\right)^{\mathbf{1}_{\{k_1=0\}}}}{\prod_{k_1<k<k_2}(z_k-z_{k+1})\,(z_{k_1+1}-\zeta_1)}.$$

The contours around 1 are ordered according to $\vec{\varepsilon}$.

Under KPZ scaling the indices i and j are re-scaled as $i = n_{r^*} + v_T u$ and $j = n_{k_2} + v_T v$, where we ignore rounding. Note that $v \leqslant 0$ since $s = k_2 < p$. We have that

$$\begin{aligned}(i-n_{k_1}, \Delta_{k_1,r^*}(m,a)) &= \Delta_{k_1,r^*}(n,m,a) + c_0\big(u/(\Delta_{k_1,r^*}t)^{1/3},0,0\big)\cdot(\Delta_{k_1,r^*}t\,T)^{1/3}, \\ (j-n_{k_2-1}, \Delta_{k_2}(m,a)) &= \Delta_{k_2}(n,m,a) + c_0\big(v/(\Delta_{k_2}t)^{1/3},0,0\big)\cdot(\Delta_{k_2}t\,T)^{1/3}.\end{aligned}$$

The triple $(i-n_{k_1}, \Delta_{k_1,r^*}(m,a))$ has the form (5.18) and $(j-n_{k_2-1}, \Delta_{k_2}(m,a))$ has the form (5.19).

Now we choose contours for the variables. We choose γ_{τ_1} to be $w_0(\hat{\sigma}_1, d(u))$ with $K := \Delta_{k_1,r^*}t\,T$. Then with an appropriate choice of $d(u)$ from Lemma 5.3, we have the estimate (5.11):

$$|G(\zeta_1(\hat{\sigma}_1) \,|\, (i-n_{k_1}, \Delta_{k_1,r^*}(m,a)))|^{-1} \leqslant C_3 e^{-C_4\hat{\sigma}_1^2+\Psi\left(u/(\Delta_{k_1,r^*}t)^{1/3}\right)}.$$

Next we choose $\gamma_{R_{k_2}}(1)$, the contour of z_{k_2}, to be $w_1(\sigma_{k_2}, D(v))$ with $K := \Delta_{k_2} t\, T$ so that we get the estimate (5.12):

$$|G(z_{k_2}(\sigma_{k_2}) \,|\, (j - n_{k_2-1} - 1, \Delta_{k_2}(m, a)))| \leqslant C_3 e^{-C_4 \sigma_{k_2}^2 + \Psi\left(v/(\Delta_{k_2} t)^{1/3}\right)}.$$

For $k_1 < k < k_2$, we choose the contour $\gamma_{R_k}(1)$ to be $w_1(\sigma_k, D_k)$ with $K := \Delta_k t\, T$ such that we have the estimate (5.12) from Lemma 5.3:

$$|G(z_k(\sigma_k) \,|\, \Delta_k(n, m, a))| \leqslant C_3 e^{-C_3 \sigma_k^2}.$$

The parameter D_k may be chosen from the range $[C_1, C_2(\Delta_k t\, T)^{1/3}]$. We have seen that we can choose these D_ks such that they are ordered according to ε. The parameter D_{k_2-1} has to be ordered with respect to $D(v)$. We can first chose these two and then choose the remaining D_ks accordingly.

To see that D_{k_2-1} and $D(v)$ can be ordered, set $\Delta_1 = (\Delta_{k_2-1})^{1/3}$ and $\Delta_2 = (\Delta_{k_2} t)^{1/3}$. Since $v \leqslant 0$, $D(v)$ may be chosen such that $D(v)/\Delta_2$ belongs to the range $[C_1/\Delta_2 + \delta(v)_-^{1/2}/\Delta_2^{3/2}, C_2 T^{1/3}]$. The number D_{k_2-1}/Δ_1 may belong to $[C_1/\Delta_1, C_2 T^{1/3}]$. If $\varepsilon_{k_2-1} = 2$ then we require $D_{k_2-1}/\Delta_1 < (D(v)-1)/\Delta_2$, say, and this is possible within the aforementioned ranges. Suppose $\varepsilon_{k_2-1} = 1$. Then we are fine so long as $\delta(v)_-^{1/2} < C_2 \Delta_2^{3/2} T^{1/3} - C_1 \Delta_2^{1/2}$. Now since $j \in (n_{k_2-1}, n_{k_2}]$, we have $(v)_- \leqslant \Delta_{k_2} n / v_T \leqslant (\Delta_{k_2} t/c_0)\, T^{2/3} + C_{q,L} T^{1/3}$. So $(v)_-^{1/2} \leqslant (\Delta_{k_2} t/c_0)^{1/2}\, T^{1/3} + C_{q,L} T^{1/6}$. Therefore, it suffices to have $\delta < C_2 \Delta_2^{3/2} (\Delta_{k_2} t/c_0)^{-1/2}$, which is the case since δ satisfies (5.13).

Let $F_T(r, u; s, v)$ be the re-scaling of our matrix by (5.4). Having chosen the contours, the estimates above imply the following, if we argue as in the proof of goodness of $L^{\vec{\varepsilon}}[k_1, k_2 \,|\, (k_1, k_2]]$.

$$\begin{aligned}
|F_T(r, u; s, v)| &\leqslant C_{q,L}\, v_T\, (T^{-\frac{1}{3}})^{k_2 - k_1 + 1} \int_{\mathbb{R}^{k_2-k_1+1}} d\hat{\sigma}_1 \cdots \sigma_{k_2}\, e^{-C_4(\hat{\sigma}_1^2 + \cdots + \sigma_{k_2}^2)} (T^{\frac{1}{3}})^{k_2-k_1} \times \\
&\quad \mathbf{1}_{\{v \leqslant 0\}} e^{-\mu u + \Psi\left(u/(\Delta_{k_1, r^*} t)^{1/3}\right)}\, e^{\mu v + \Psi\left(v/(\Delta_{k_2} t)^{1/3}\right)} \\
&\leqslant C_{q,L}\, e^{-\mu u + \Psi\left(u/(\Delta_{k_1, r^*} t)^{1/3}\right)} \cdot \mathbf{1}_{\{v \leqslant 0\}} e^{\mu v + \Psi\left(v/(\Delta_{k_2} t)^{1/3}\right)}.
\end{aligned}$$

Both $\Delta_{k_1, r^*} t$ and $\Delta_{k_2} t$ are at least $\min_k \{\Delta_k t\}$ and μ satisfies (5.13). So the functions of u and v above are bounded and integreble by (5.15), and the matrix is good.

For the proof of convergence of $L^{\vec{\varepsilon}}[k_1 \,|\, (k_1, k_2]]$ to $(-1)^{k_2 - k_1} F^{\vec{\varepsilon}}[k_1 \,|\, (k_1, k_2]]$ we can repeat the argument for convergence of $L^{\vec{\varepsilon}}[k_1, k_2 \,|\, (k_1, k_2]]$. ∎

Proof of Lemma 5.6. Since $L^{\vec{\varepsilon}}[k_1 \,|\, (k_1, k_2]]$ has non-zero blocks only on column block k_2,

$$L^{\vec{\varepsilon}}[k_1 \,|\, (k_1, k_2]] \cdot B(r, i; s, j) = (1 + \Theta(k_2 \,|\, s)) \sum_{\ell \in (n_{k_2-1}, n_{k_2}]} L^{\vec{\varepsilon}}[k_1 \,|\, (k_1, k_2]](r, i; k_2, \ell)\, B(k_2, \ell, s, j).$$

We can compute this using Lemma 5.9 as follows.

$$
\begin{aligned}
&L^{\vec{\varepsilon}}[k_1\,|\,(k_1,k_2]]\cdot B(r,i;s,j) = (1+\Theta(k_2\,|\,s))\,c(r,i;s,j)\times\\
&\mathbf{1}_{\{k_1<r^*,s<k_2<p\}}\frac{1}{w_c^2}\oint_{\gamma_{\tau_1}} d\zeta_1 \oint_{\gamma_{R_{k_2}}(1)}\oint_{\gamma_{\tau_2}} d\zeta_2\,\frac{f(\zeta_1,z_{k_2})}{G\big(\zeta_1\,|\,i-n_{k_1},\Delta_{k_1,r^*}(m,a)\big)}\times\\
&\sum_{\ell\in(n_{k_2-1},n_{k_2}]}\frac{1}{G\big(z_{k_2}\,|\,n_{k_2-1}-\ell+1,-\Delta_{k_2}(m,a)\big)\,G\big(\zeta_2\,|\,\ell-j+1,\Delta_{s,k_2}(m,a)\big)}\\
&= (1+\Theta(k_2\,|\,s))\,c(r,i;s,j)\,\mathbf{1}_{\{k_1<r^*,s<k_2<p\}}\frac{1}{w_c}\oint_{\gamma_{\tau_1}} d\zeta_1\oint_{\gamma_{\tau_2}} d\zeta_2\oint_{\gamma_{R_{k_2}}(1)}\\
&\Big[\frac{f(\zeta_1,z_{k_2})(z_{k_2}-\zeta_1)^{-1}G\big(z_{k_2}\,|\,\Delta_{k_2}(n,m,a)\big)}{G\big(\zeta_1\,|\,i-n_{k_1},\Delta_{k_1,r^*}(m,a)\big)\,G\big(\zeta_2\,|\,n_{k_2}-j+1,\Delta_{s,k_2}(m,a)\big)}-\\
&\frac{f(\zeta_1,z_{k_2})(z_{k_2}-\zeta_1)^{-1}G\big(z_{k_2}\,|\,0,\Delta_{k_2}(m,a)\big)}{G\big(\zeta_1\,|\,i-n_{k_1},\Delta_{k_1,r^*}(m,a)\big)\,G\big(\zeta_2\,|\,n_{k_2-1}-j+1,\Delta_{s,k_2}(m,a)\big)}\Big]\\
&= (1+\Theta(k_2\,|\,s))\Big[L^{\vec{\varepsilon}}[k_1,k_2,|\,(k_1,k_2]] - \mathcal{S}L^{\vec{\varepsilon}}[k_1,k_2,|\,(k_1,k_2]]\Big].
\end{aligned}
$$

The function f is from (5.23). We observed above that $f(\zeta_1,z_d)(z_{k_2}-\zeta_2)^{-1}G(z_{k_2}\,|\,\Delta_{k_2}(n,m,a))$ divided by $G(\zeta_1\,|\,\cdots)\cdot G(\zeta_2\,|\,\cdots)$ makes the integrand of $L^{\vec{\varepsilon}}[k_1,k_2,|\,(k_1,k_2]]$, as is required.

To complete the proof we show that the matrix $\mathcal{S}L^{\vec{\varepsilon}}[k_1,k_2,|\,(k_1,k_2]]$ is small. The argument is analogous to the prior proof of smallness of $\mathcal{S}L^{\vec{\varepsilon}}[k_1,k_2,k_3\,|\,(k_1,k_2]]$. The role of variables ζ_1,ζ_2,ζ_3 from there is now given to ζ_1,z_{k_2},ζ_2, respectively. The parameter $\lambda=\Delta_{k_2}n/\nu_T=(\Delta_{k_2}t/c_0)T^{2/3}+C_{q,L}T^{1/3}$. Since the ζ_2-contour lies around 0 and the z_{k_2}-contour around 1, there is no ordering between them. We need the z_{k_2}-contour to be ordered with respect to the z_{k_2-1}-contour according to ε_{k_2-1}, and for this we may repeat the prior argument for the goodness of $L^{\vec{\varepsilon}}[k_1,|\,(k_1,k_2]]$.

After choosing contours as before we get the following estimates for the G-functions.

$$
\begin{aligned}
&|G\big(\zeta_1(\sigma_1)\,|\,\Delta_{k_1,r^*}n+\nu_T u,\Delta_{k_1,r^*}(m,a)\big)|^{-1}\leqslant C_3e^{-C_4\sigma_1^2+\Psi(u/\Delta_1)}\\
&|G\big(\zeta_2\,|\,\Delta_{s,k_2}n-\nu_T(\nu+\lambda),\Delta_{s,k_2}(m,a)\big)|^{-1}\leqslant C_3e^{-C_4\sigma_2^2+\Psi\left(-(\nu+\lambda)/\Delta_2\right)}\\
&|G\big(z_{k_2}(\sigma_3)\,|\,\Delta_{k_2}n-\nu_T\lambda,\Delta_{k_2}(m,a)\big)|\leqslant C_3e^{-C_4\sigma_3^2+\Psi(-\lambda/\Delta_3)}.
\end{aligned}
$$

Here, $\Delta_1=(\Delta_{k_1,r^*}t)^{1/3}, \Delta_2=(\Delta_{s,k_2}t)^{1/3}$ and $\Delta_3=(\Delta_{k_2}t)^{1/3}$.

Using these estimates, and arguing as before, we find the following estimate for the re-scaled kernel F_T of $\mathcal{S}L^{\vec{\varepsilon}}[k_1,k_2,|\,(k_1,k_2]]$.

$$
|F_T(r,u;s,\nu)|\leqslant \underbrace{C_{q,L}e^{-\mu\lambda+\Psi(-\lambda/\Delta_3)}}_{\eta_T}\cdot e^{-\mu u+\Psi(u/\Delta_1)}\cdot e^{\mu(\nu+\lambda)+\Psi(-(\nu+\lambda)/\Delta_2)}.
$$

We observe that $\eta_T=C_{q,L}e^{-\mu\lambda-\mu_1(\lambda/\Delta_3)^{3/2}}\to 0$, and the two functions of u and ν are bounded and integrable over $\mathbb{R}$ due to (5.15). So the matrix is small. ■

5.3.3. *Proof of claims regarding* $L[k,k\,|\,\emptyset]$. First we will prove that $L[k,k\,|\,\emptyset]$ is good and convergent to $F[k,k\,|\,\emptyset]$. Then we will prove Lemma 5.7 by first showing that $L[k_1,k_1,k_2\,|\,\emptyset]$ is good and convergent, and then that $\mathcal{S}L[k_1,k_1,k_2\,|\,\emptyset]$ is small.

Proof that $L[k,k\,|\,\emptyset]$ **is good and convergent**. The matrix $L[k,k\,|\,\emptyset]$ has non-zero blocks (r,s) only if $s<k<r^*$. Let us fix such k,r and s, so then $L[k,k\,|\,\emptyset](r,i;s,j)$ equals

$$c(r,i;s,j)\,\frac{1}{w_c}\oint_{\gamma_{\tau_1}} d\zeta_1 \oint_{\gamma_{\tau_2}} d\zeta_2\,\frac{(\zeta_1-\zeta_2)^{-1}}{G\big(\zeta_1\,|\,i-n_k,\Delta_{k,r^*}(m,a)\big)\,G\big(\zeta_2\,|\,n_k-j+1,\Delta_{s,k}(m,a)\big)}.$$

Ignoring rounding, the indices are re-scaled according to $i=n_{r^*}+\nu_T u$ and $j=n_s+\nu_T v$. Note that $v\leqslant 0$ since $s<p$. In this case the KPZ re-scaling of $(i-n_k,\Delta_{k,r^*}(m,a))$ looks like (5.18), and that of $(n_k-j,\Delta_{s,k}(m,a))$ like (5.19). Set $\Delta_1=(\Delta_{k,r^*}t)^{1/3}$ and $\Delta_2=(\Delta_{s,k}t)^{1/3}$.

For establishing goodness, contours are chosen so that the ζ_1-contour is $w_0(\sigma_1,d(u))$ with $K:=\Delta_{k,r^*}t\,T$. The ζ_2-contour is $w_0(\sigma_2,d(-v))$ with $K:=\Delta_{s,k}t\,T$. With appropriate choices for $d(u)$ and $d(-v)$, Lemma 5.3 provides the estimates

$$|G(\zeta_1(\sigma_1)\,|\,\Delta_{k,r^*}n+\nu_T u,\Delta_{k,r^*}(m,a))|^{-1}\leqslant C_3\,e^{-C_4\sigma_1^2+\Psi(u/\Delta_1)},$$
$$|G(\zeta_2(\sigma_2)\,|\,\Delta_{s,k}n-\nu_T v,\Delta_{s,k}(m,a))|^{-1}\leqslant C_3\,e^{-C_4\sigma_2^2+\Psi(-v/\Delta_2)}.$$

We need to have $\tau_2<\tau_1$, which translates to $d(u)/\Delta_1<(d(-v)-1)/\Delta_2$, say. Since $v\leqslant 0$, the number $d(-v)/\Delta_2$ may be chosen from $[C_1/\Delta_2,C_2T^{1/3}]$ once T is large enough it terms of q and L. When $u\geqslant 0$, $d(u)/\Delta_1$ can be chosen from $[C_1/\Delta_1,C_2T^{1/3}]$, and we can order the contours accordingly. If $u\leqslant 0$ then $d(u)/\Delta_1$ may belong to $[C_1/\Delta_1+\delta(u)_-^{1/2}/\Delta_1^{3/2},C_2T^{1/3}]$. We can order the contours so long as $\delta(u)_-^{1/2}<C_2\Delta_1^{3/2}\,T^{1/3}-C_1\Delta_1^{1/2}$. We have that $(u)_-\leqslant(\Delta_{r^*})t/c_0)T^{2/3}+C_{q,L}T^{1/3}$. Therefore, as before, we are fine since δ satisfies (5.13).

Let F_T be the re-scaled kernel of $L[k,k\,|\,\emptyset]$ by (5.4). The estimates above for the G-functions and the same argument used to show goodness of $L^{\vec{\varepsilon}}[k_1,k_2\,|\,(k_1,k_2]]$ implies the following bound.

$$|F_T(r,u;s,v)|\leqslant C_{q,L}\,e^{-\mu u+\Psi(u/\Delta_1)}\cdot e^{\mu v+\Psi(-v/\Delta_2)}. \tag{5.24}$$

This certifies goodness of $L[k,k\,|\,\emptyset]$ by (5.15).

The proof of convergence to $F[k,k\,|\,\emptyset]$ is same as that of $L^{\vec{\varepsilon}}[k_1,k_2\,|\,(k_1,k_2]]$ converging to the kernel $(-1)^{k_2-k_1}F^{\vec{\varepsilon}}[k_1,k_2\,|\,(k_1,k_2]]$. So we omit the details. ∎

Proof of Lemma 5.7. We multiply $L[k,k \mid \emptyset]$ by B using Lemma 5.9.

$$L[k,k \mid \emptyset] \cdot B\,(r,i;s,j) = \sum_{k_2} (1+\Theta(k_2 \mid s))\, \mathbf{1}_{\{k<r^*,\, s<k_2<k\}}\, c(r,i;s,j) \times$$

$$\frac{1}{w_c^2} \oint_{\gamma_{\tau_1}} d\zeta_1 \oint_{\gamma_{\tau_2}} d\zeta_2 \oint_{\gamma_{\tau_3}} d\zeta_3 \frac{(\zeta_1-\zeta_2)^{-1}}{G(\zeta_1 \mid i-n_k, \Delta_k(m,a))} \times$$

$$\Big[\sum_{\ell \in (n_{k_2-1}, n_{k_2}]} \frac{1}{G\big(\zeta_2 \mid n_k-\ell+1, \Delta_k(m,a)\big) G\big(\zeta_3 \mid \ell-j+1, \Delta_{s,k_2}(m,a)\big)}\Big]$$

$$= \sum_{k_2} (1+\Theta(k_2 \mid s)) \cdot \big[L[k,k,k_2 \mid \emptyset](r,i;s,j) - (\mathcal{S}L)[k,k,k_2 \mid \emptyset](r,i;s,j)\big].$$

Now consider $L[k_1,k_1,k_2 \mid \emptyset]$ to see that it is good, and converges to $F[k_1,k_1,k_2 \mid \emptyset]$. Recall

$$L[k_1,k_1,k_2 \mid \emptyset](r,i;s,j) = \mathbf{1}_{\{k_1<r^*,\, s<k_2<k_1\}}\, c(r,i;s,j) \frac{1}{w_c} \oint_{\gamma_{\tau_1}} d\zeta_1 \oint_{\gamma_{\tau_2}} d\zeta_2 \oint_{\gamma_{\tau_3}} d\zeta_3$$

$$\frac{(\zeta_1-\zeta_2)^{-1}(\zeta_2-\zeta_3)^{-1}}{G\big(\zeta_1 \mid i-n_{k_1}, \Delta_{k_1,r^*}(m,a)\big)\, G\big(\zeta_2 \mid \Delta_{k_2,k_1}(n,m,a)\big)\, G\big(\zeta_1 \mid n_{k_2}-j+1, \Delta_{s,k_2}(m,a)\big)}.$$

This matrix has the same structure as $L[k,k \mid \emptyset]$, and the proof of goodness and convergence is analogous. The new terms in the integrand are $(\zeta_2-\zeta_3)^{-1}$ and $G(\zeta_2 \mid \Delta_{k_2,k_1}(n,m,a))$. The latter converges to $\mathcal{G}(\zeta_2 \mid \Delta_{k_2,k_1}(t,x,\xi))$ under KPZ re-scaling by Lemma 5.2, which leads to the limit kernel $F[k_1,k_1,k_2 \mid \emptyset]$. In the proof of goodness, one uses estimate (5.11) from Lemma 5.3 to derive the same bound (5.24) on the re-scaled kernel of $L[k_1,k_1,k_2 \mid \emptyset]$.

During the estimates leading to goodness, one has to ensure that the contours are ordered appropriately. We require that $\tau_2 < \tau_1, \tau_3$ due to the term $(\zeta_1-\zeta_2)^{-1}(\zeta_2-\zeta_3)^{-1}$. We choose the ζ_2-contour to be $w_0(\sigma_2, d_2)$ with $K := \Delta_{k_2,k_2} t\, T$. The parameter d_2 may be chosen from an interval with length of order $T^{1/3}$. Then, the same argument used for ordering contours in showing goodness of $L[k,k \mid \emptyset]$ shows that contours can be ordered accordingly.

We are left to prove that $\mathcal{S}L[k_1,k_1,k_2 \mid \emptyset]$ is small. It is similar to proofs of smallness so far. Let us fix k_1, k_2 and consider a non-zero (r,s)-block, so then $k_1 < r^*$ and $s < k_2 < k_1$. We have

$$\mathcal{S}L[k_1,k_1,k_2 \mid \emptyset](r,i;s,j) = c(r,i;s,j)\, \frac{1}{w_c} \oint_{\gamma_{\tau_1}} d\zeta_1 \oint_{\gamma_{\tau_2}} d\zeta_2 \oint_{\gamma_{\tau_3}} d\zeta_3$$

$$\frac{(\zeta_1-\zeta_2)^{-1}(\zeta_2-\zeta_3)^{-1}}{G\big(\zeta_1 \mid i-n_{k_1}, \Delta_{k_1,r^*}(m,a)\big)\, G\big(\zeta_2 \mid n_{k_1}-n_{k_2-1}, \Delta_{k_2,k_1}(m,a)\big)\, G\big(\zeta_3 \mid n_{k_2-1}-j+1, \Delta_{s,k_2}(m,a)\big)}.$$

The radii satisfy $\tau_2 < \tau_1, \tau_3 < 1-\sqrt{q}$.

We have $i = n_{r^*} + \nu_T u$ and $j = n_s + \nu_T v$. Set $\lambda = \Delta_{k_2} n/\nu_T = (\Delta_{k_2} t/c_0) T^{2/3}$. Also set $\Delta_1 = (\Delta_{k_1,r^*} t)^{1/3}$, $\Delta_2 = (\Delta_{k_2,k_1} t)^{1/3}$ and $\Delta_1 = (\Delta_{s,k_2} t)^{1/3}$. Then,

$$(i-n_{k_1}, \Delta_{k_1,r^*}(m,a)) = (\Delta_{k_1,r^*} n + \nu_T u, \Delta_{k_1,r^*}(m,a)),$$
$$(n_{k_1}-n_{k_2-1}, \Delta_{k_2,k_1}(m,a)) = (\Delta_{k_2,k_1} n + \nu_T \lambda, \Delta_{k_2,k_1}(m,a)),$$
$$(n_{k_2-1}-j+1, \Delta_{s,k_2}(m,a)) = (\Delta_{s,k_2} n - \nu_T(v+\lambda), \Delta_{s,k_2}(m,a)).$$

We choose the ζ_1-contour to be $w_0(\sigma_1, d(u))$, the ζ_2-contour as $w_0(\sigma_2, d(\lambda))$ and the ζ_3-contour as $w_0(\sigma_3, d(-\nu-\lambda))$. The corresponding values of K are $\Delta_{k_1,r^*} t\,T$, $\Delta_{k_2,k_1} t\,T$ and $\Delta_{s,k_2} t\,T$, respectively. By Lemma 5.3, we have the following estimates.

$$|G(\zeta_1(\sigma_1)\,|\,\Delta_{k_1,r^*} n + \nu_T u, \Delta_{k_1,r^*}(m,a))|^{-1} \leqslant C_3\, e^{-C_4\sigma_1^2 + \Psi(u/\Delta_1)},$$
$$|G(\zeta_2(\sigma_2)\,|\,\Delta_{k_2,k_1} n + \nu_T \lambda, \Delta_{k_2,k_1}(m,a))|^{-1} \leqslant C_3\, e^{-C_4\sigma_2^2 + \Psi(\lambda/\Delta_2)},$$
$$|G(\zeta_3\,|\,\Delta_{s,k_2} n - \nu_T(\nu+\lambda), \Delta_{s,k_2}(m,a))|^{-1} \leqslant C_3\, e^{-C_4\sigma_3^2 + \Psi(-(\nu+\lambda)/\Delta_3)}.$$

To ensure constraints on the radii of contours, we need $(d(\lambda)-1)/\Delta_2 > \max\{d(u)/\Delta_1, d(-\nu-\lambda)/\Delta_3\}$, say. We can choose $d(\lambda)/\Delta_2$ from the interval $[C_1/\Delta_2, C_2 T^{1/3}]$. We also have $(u)_- \leqslant \Delta_{r^*} n/\nu_T$, and the square root of the latter is of order $T^{1/3}$. Since $\nu \leqslant 0$ (due to $s < p$), $\nu + \lambda \leqslant \lambda$, and $\lambda^{1/2}$ is of order $T^{1/3}$. Then, since δ satisfies (5.13), arguing as before we see that the ds can be chosen to satisfy the constraints.

Let F_T be the re-scaled kernel of $\mathcal{S}L[k_1,k_1,k_2\,|\,\emptyset]$ by (5.4). Using the estimates above and arguing as before we find the following.

$$|F_T(r,u;s,\nu)| \leqslant C_{q,L}\, e^{\mu(\nu-u)}\, e^{\Psi(u/\Delta_1)+\Psi(\lambda/\Delta_2)+\Psi((-\nu-\lambda)/\Delta_3))}$$
$$= \underbrace{C_{q,L}\, e^{-\mu\lambda+\Psi(\lambda/\Delta_2)}}_{\eta_T} \cdot e^{-\mu u + \Psi(u/\Delta_1)} \cdot e^{\mu(\nu+\lambda)+\Psi((-\nu-\lambda)/\Delta_3)}.$$

We observe that $\eta_T = C_{q,L} e^{(\frac{\mu_2}{\Delta_2}-\mu)\lambda}$ tends to zero since μ satisfies (5.13). The functions of u and ν are bounded and integrable over $\mathbb{R}$. So the matrix is small. ∎

5.3.4. *Proof of claims regarding* $L[p\,|\,p]$. First we will prove that $L[p\,|\,p]$ is good with limit $-F[p\,|\,p]$, which will complete the proof of Proposition 5.1. Then we will prove Lemma 5.8.

Proof that it is good and convergent. The argument is similar to the goodness and convergence of $L[k_1\,|\,(k_1,k_2]]$ as these matrices are alike. The only non-zero row block of $L[p\,|\,p]$ is for $r = p$ (see L_p from Lemma 4.6). On the (p,s)-block the indices i,j are re-scaled as $i = n_{p-1} + \nu_T u$ for $0 \leqslant u \leqslant \Delta_p n/\nu_T$, and $j = n_{s*} + \nu_T \nu$. We ignore rounding. So we find that

$$(n_p - i, \Delta_p(m,a)) = \Delta_p(n,m,a) + c_0\big(-u/(\Delta_p t)^{1/3}, 0, 0\big)\cdot(\Delta_p t\, T)^{1/3}$$
$$(n_p - j, \Delta_{s^*,p}(m,a)) = \Delta_{s^*,p}(n,m,a) + c_0\big(-\nu/(\Delta_{s^*,p} t)^{1/3}, 0, 0\big)\cdot(\Delta_{s^*,p} t\, T)^{1/3}.$$

We choose γ_{τ_2} to be the contour $w_0(\sigma_1, d(-\nu))$ with $K := \Delta_{s^*,p} tT$ and $\gamma_{R_p}(1)$ to be the contour $w_1(\sigma_2, d(-u))$ with $K := \Delta_p tT$. Since the ζ_2-contour is around 0 and the z_p-contour is around 1, we can ensure that $|z_p - \zeta_2| \geqslant C_{q,L} T^{-1/3}$ along these contours. According to Lemma 5.3 we then have the following estimates.

$$|G(\zeta_2(\sigma_1)\,|\,n_p - j + 1, \Delta_{s^*,p}(m,a))|^{-1} \leqslant C_3 e^{-C_4\sigma_1^2 + \Psi(-\nu/(\Delta_{s^*,p} t)^{1/3})}, \tag{5.25}$$
$$|G(z_p(\sigma_2)\,|\,n_p - i, \Delta_p(m,a))| \leqslant C_3 e^{-C_4\sigma_2^2 + \Psi(-u/(\Delta_p t)^{1/3})}.$$

The re-scaled kernel of $L[p\,|\,p]$ according to (5.4) then satisfies the following, arguing as before.

$$|F_T(r,u;s,\nu)| \leqslant \mathbf{1}_{\{r=p\}} C_{q,L}\, \mathbf{1}_{\{u\geqslant 0\}} e^{-\mu u + \Psi(-u/(\Delta_p t)^{1/3})} \cdot e^{\mu\nu + \Psi(-\nu/(\Delta_{s^*,p} t)^{1/3})}.$$

The functions of u and v above are bounded and integrable by (5.15). So $L[p\,|\,p]$ is good. The argument for convergence of $L[p\,|\,p]$ to $-F[p\,|\,p]$ is the same as before.

Proof of Lemma 5.8. We multiply $L[p\,|\,p]$ by B using Lemma 5.9:

$$L[p\,|\,p]\cdot B(r,i;s,j) = \sum_{k=1}^{p} (1+\Theta(k\,|\,s))\big(\hat{L}_k - (\mathcal{S}\hat{L})_k\big)(r,i;s,j),$$

where

$$\hat{L}_k(r,i;s,j) = \mathbf{1}_{\{r=p,\, s<k^*\}}\, c(r,i;s,j)\,\frac{1}{w_c}\oint_{\gamma_{\tau_2}} d\zeta_2 \oint_{\gamma_{\tau_3}} d\zeta_3 \oint_{\gamma_{R_p}(1)} dz_p$$
$$\frac{G\big(z_p\,|\,n_p-i,\Delta_p(m,a)\big)(z_p-\zeta_2)^{-1}(\zeta_2-\zeta_3)^{-1}}{G\big(\zeta_2\,|\,n_p-n_k,\Delta_{k^*,p}(m,a)\big)G\big(\zeta_3\,|\,n_k-j+1,\Delta_{s,k^*}(m,a)\big)},$$

and $(\mathcal{S}\hat{L})_k$ looks the same as $\hat{L}_k$ except for n_k being changed to n_{k-1} in both of $G(\zeta_2\,|\,n_p-n_k,\cdots)$ and $G(\zeta_3\,|\,n_k-j+1,\cdots)$ above. The contours are arranged to satisfy $\tau_2<\tau_3<w_c$.

Now if $k<p$ then we see in the above that $\hat{L}_k$ equals $L[p,k\,|\,p]$ as $k^*=k$. However, when $k=p$, $\hat{L}_p=0$ because there is no pole at $\zeta_2=0$ in its integrand due to $n_p=n_k$ and the ζ_2-contour being the innermost one. So in this way we get the matrices $L[p,k\,|\,p]$. Now consider the matrix $(\mathcal{S}\hat{L})_k$. If $k<p$ then it equals $(\mathcal{S}L)[p,k\,|\,p]$ by definition. When $k=p$ it is actually $(\mathcal{S}L)[p,p-1\,|\,p]$ by definition since k^* then equals $p-1$. This implies the expression for $L[p\,|\,p]\cdot B$ given in the lemma.

The goodness and convergence of $L[p,k\,|\,p]$ is analogous to that for $L[p\,|\,p]$ above. We explain the difference. We use the estimates from (5.25) to estimate the G-functions associated to the ζ_3 and z_p contours. They involve the variables u and v from the kernel. There is an additional function $G(\zeta_2\,|\,\Delta_p(n,m,a))$ in the denominator of the integrand. For it we choose the ζ_2-contour to be $w_0(\sigma,d)$ with $K=\Delta_{k,p}tT$, and use the estimate (5.11) from Lemma 5.3. We have to keep the ζ_2 and ζ_3 contours ordered ($\tau_2<\tau_3$), for which we require $d/(\Delta_{k,p}t)^{1/3} > (d(-v)+1)/(\Delta_{s^*,p}t)^{1/3}$. This is ensured as before since the parameter d may be chosen from an interval whose length is of order $T^{1/3}$.

The proof of smallness of $(\mathcal{S}L)[p,k\,|\,p]$ is also similar to the smallness of $(\mathcal{S}L)[k_1,k_2\,|\,(k_1,k_2])]$ from before. Arguing as there, we will get the following estimate for the re-scaled kernel $F_T(r,u;s,v)$ of $(\mathcal{S}L)[p,k\,|\,p]$. Set $\lambda=\Delta_k n/\nu_T$ and $\eta_T = e^{-\mu\lambda+\Psi(\lambda/(\Delta_{k,p}t)^{1/3})}$. Recall $1\leqslant k<p$, so $\lambda\to\infty$ and $\Delta_{k,p}t>0$. If μ satisfies (5.13), then $\eta_T\to 0$ and

$$|F_T(r,u;s,v)| \leqslant \mathbf{1}_{\{r=p\}} C_{q,L}\,\eta_T\,\mathbf{1}_{\{u\geqslant 0\}} e^{-\mu u+\Psi(-u/(\Delta_p t)^{1/3})}\cdot e^{\mu(v+\lambda)+\Psi\left(-(v+\lambda)/(\Delta_{s^*,k}t)^{1/3}\right)},$$

which guarantees smallness. ∎

5.4. **Tying up loose ends.** Here we will prove Proposition 5.3 and that the limit from Theorem 1 is a probability distribution.

Proof of Proposition 5.3. It is enough to show $L\cdot B_2$ is small where L is any one of the matrices $L[k,k\,|\,\emptyset]$, $L[k_1,k_1,k_2\,|\,\emptyset]$, $L[p\,|\,p]$, $L[p,k\,|\,p]$, $L^{\vec{\varepsilon}}[k_1,k_2\,|\,(k_1,k_2]]$, $L^{\vec{\varepsilon}}[k_1\,|\,(k_1,k_2]]$ or $L^{\vec{\varepsilon}}[k_1,k_2,k_3\,|\,(k_1,k_2]]$. Recall from Lemma 5.4 that B_2 is a weighted sum of the matrices $(\mathcal{S}L)[k,k\,|\,\emptyset]$. So it suffices to prove that each of the aforementioned matrices are small when the multiplication by B_2 is replaced by $(\mathcal{S}L)[k,k\,|\,\emptyset]$.

Lemma 5.10. *Consider the matrix* $\mathcal{S}L[k,k\,|\,\emptyset]$ *and denote* $F_{T,k}$ *its re-scaled kernel according to* (5.4). *Set* $\lambda_k = \Delta_k n/\nu_T = (\Delta_k t/c_0)\,T^{2/3} + C_{q,L}T^{1/3}$, $\Delta_1 = (\Delta_{k,r^*}t)^{1/3}$ *and* $\Delta_2 = (\Delta_{s,k}t)^{1/3}$. *The following bound holds for* $F_{T,k}$.

$$|F_{T,k}(r,u;s,v)| \leqslant \mathbf{1}_{\{s<k<r^*\}}\, C_{q,L}\, e^{-\mu(u+\lambda_k)+\Psi\big((u+\lambda_k)/\Delta_1\big)} \cdot e^{\mu(v+\lambda_k)+\Psi\big(-(v+\lambda_k)/\Delta_2\big)}.$$

Proof. Let us recall $\mathcal{S}L[k,k\,|\,\emptyset]$ from Lemma 5.4. The entry $\mathcal{S}L[k,k\,|\,\emptyset](r,i;s,j)$ equals

$$\mathbf{1}_{\{s<k<r^*\}}\frac{c(r,i;s,j)}{w_c}\oint_{\gamma_{\tau_1}} d\zeta_1 \oint_{\gamma_{\tau_2}} d\zeta_2 \frac{(\zeta_1-\zeta_2)^{-1}}{G\big(\zeta_1\,|\,i-n_{k-1},\Delta_{k,r^*}(m,a)\big)\,G\big(\zeta_2\,|\,n_{k-1}-j+1,\Delta_{s,k}(m,a)\big)}.$$

Indices i,j are re-scaled according to (5.17). Ignoring the rounding, this means that

$$(i-n_{k-1},\Delta_{k,r^*}(m,a)) = \Delta_{k,r^*}(n,m,a) + c_0\big((u+\lambda_k)/\Delta_1,0,0\big)\cdot(\Delta_{k,r^*}t\,T)^{1/3},$$
$$(n_{k-1}-j,\Delta_{s,k}(m,a)) = \Delta_{s,k}(n,m,a) - c_0\big((v+\lambda_k)/\Delta_2,0,0\big)\cdot(\Delta_{s,k}t\,T)^{1/3}.$$

We choose the ζ_1-contour to be $w_0(\sigma_1,d(u+\lambda_k))$ with $K \coloneqq \Delta_{k,r^*}t\,T$. Similarly, the ζ_2-contour is $w_0(\sigma_2,d(-v-\lambda_k))$ with $K \coloneqq \Delta_{s,k}t\,T$. Due to the constraint $\tau_2<\tau_1$ we should have $d(u+\lambda_k)/\Delta_1 < (d(-v-\lambda_k)-1)/\Delta_2$. In this case, $|\zeta_1(\sigma_1)-\zeta_2(\sigma_2)|^{-1} \leqslant C_{q,L}T^{1/3}$. Furthermore, with $d(\cdot)$s chosen according to Lemma 5.3 we have the following estimates.

$$|G(\zeta_1(\sigma_1)\,|\,i-n_{k-1},\,\Delta_{k,r^*}(m,a))|^{-1} \leqslant C_3 e^{-C_4\sigma_1^2+\Psi\big((u+\lambda_k)/\Delta_1\big)},$$
$$|G(\zeta_2(\sigma_2)\,|\,n_{k-1}-j+1,\,\Delta_{s,k}(m,a))|^{-1} \leqslant C_3 e^{-C_4\sigma_2^2+\Psi\big(-(v+\lambda_k)/\Delta_2\big)}.$$

With these estimates, changing variables $\zeta_\ell \mapsto \sigma_\ell$ and arguing as before, we see that

$$\begin{aligned}|F_{T,k}(r,u;s,v)| &\leqslant \mathbf{1}_{\{s<k<r^*\}}\, C_{q,L}\int_{\mathbb{R}^2} d\sigma_1 d\sigma_2\, e^{-C_4(\sigma_1^2+\sigma_2^2)}\times \\ &\qquad e^{\mu(v-u)}\, e^{\Psi\big((u+\lambda_k)/\Delta_1\big)}\, e^{\Psi\big(-(v+\lambda_k)/\Delta_2\big)} \\ &= \mathbf{1}_{\{s<k<r^*\}}\, C_{q,L}\, e^{-\mu(u+\lambda_k)+\Psi\big((u+\lambda_k)/\Delta_1\big)} \cdot e^{\mu(v+\lambda_k)+\Psi\big(-(v+\lambda_k)/\Delta_2\big)}.\end{aligned}$$

It remains to order the contours. We know that if T is sufficiently large in terms of q and L, then

$$d(u+\lambda_k)/\Delta_1 \in \begin{cases}[C_1\Delta_1^{-1},\,C_2T^{1/3}] & \text{if } u+\lambda_k \geqslant 0\\ [C_1\Delta_1^{-1}+\Delta_1^{-3/2}\delta(u+\lambda_k)_-^{1/2},\,C_2T^{1/3}] & \text{if } u+\lambda_k<0;\end{cases}$$
$$d(-v-\lambda_k)/\Delta_2 \in \begin{cases}[C_1\Delta_2^{-1},\,C_2T^{1/3}] & \text{if } v+\lambda_k \leqslant 0\\ [C_1\Delta_2^{-1}+\Delta_2^{-3/2}\delta(v+\lambda_k)^{1/2},\,C_2T^{1/3}] & \text{if } v+\lambda_k>0.\end{cases}$$

If $u+\lambda_k \geqslant 0$ then we can order the contours by first choosing $d(-v-\lambda_k)$ and then choosing $d(u+\lambda)$ accordingly from an interval with length of order $T^{1/3}$. Suppose $u+\lambda_k<0$. Then we will first choose $d(u+\lambda_k)$ and then $d(-v-\lambda_k)$ accordingly. We are able to do so if $C_1\Delta_1^{-1}+\Delta_1^{-3/2}\delta(u+\lambda_k)_-^{1/2} < C_2T^{1/3}$. In this regard, since $\lambda_k>0$, $(u+\lambda_k)_- \leqslant (u)_-$. Now $(u)_- \leqslant$

$\Delta_{r^*} n / \nu_T = (\Delta_{r^*} t/c_0)\, T^{2/3} + C_{q,L} T^{1/3}$. Therefore, we are fine so long as $\delta < C_2 c_0^{1/2} \Delta_1^{3/2} (\Delta_{r^*} t)^{-1/2}$, which holds because δ satisfies (5.13). ∎

Lemma 5.11. *Let $M_1, M_2, \ldots$ be a sequence of good matrices where M_n is $n \times n$ and $n = n_p$ is according to* (1.3)*. Then the sequence of matrices $M_n \cdot \mathcal{S}L[k,k\,|\,\emptyset]_{n\times n}$ is small.*

Proof. Let F_T and $F_{T,k}$ be the re-scaled kernels of M_n and $\mathcal{S}L[k,k\,|\,\emptyset]_{n\times n}$, respectively, via (5.4). Let F'_T be the one for their product. We have that

$$F'_T(r,u;s,v) = \sum_{\ell=1}^{p} \int dz\, F_T(r,u;\ell,z) F_{T,k}(\ell,z;s,v) \cdot \mathbf{1}_{\{s<k<\ell^*\}}.$$

The z-integral is over $\mathbb{R}_{<0}$ for $\ell < p$ and over $\mathbb{R}_{>0}$ if $\ell = p$. Note that $\mathcal{S}L[k,k\,|\,\emptyset]$ is non-zero only for $k < p-1$, and so we may replace ℓ^* by ℓ above. It suffices to show that for every ℓ such that $s < k < \ell$, the corresponding z-integral is a small kernel in terms of u and v.

Fix s, ℓ and k such that $s < k < \ell$. Let $g_1, \ldots, g_p$ be the bounded and integrable functions over $\mathbb{R}$ that certify goodness of F_T. Recalling Lemma 5.10, let λ denote the parameter λ_k there. Also set $\Delta_1 = (\Delta_{k,\ell} t)^{1/3}$, $\Delta_2 = (\Delta_{s,k} t)^{1/3}$ and the function $f(z) = e^{-\mu z + \Psi(z/\Delta_1)}$.

First, suppose $\ell < p$. Due to goodness of F_T and Lemma 5.10, we infer that

$$\Big| \int_{-\infty}^{0} dz\, F_T(r,u;\ell,z) F_{T,k}(\ell,z;s,v) \Big| \leqslant C_{q,L} \int_{-\infty}^{0} dz\, g_\ell(z) f(z+\lambda) \cdot g_r(u) \cdot e^{\mu(v+\lambda) + \Psi\left(-(v+\lambda)/\Delta_2\right)}.$$

By (5.15) we see that the function $e^{\mu(v+\lambda)+\Psi\left(-(v+\lambda)/\Delta_2\right)}$ is bounded and integrable over $\mathbb{R}$ in variable v. Smallness thus follows if the z-integral tends to zero as $T \to \infty$. In this regard observe that for $x \geqslant 0$, $f(x) = e^{(\frac{\mu_2}{\Delta_1} - \mu)x}$, and $\frac{\mu_2}{\Delta_1} - \mu < 0$ since μ satisfies (5.13). Therefore, $\max_{x \geqslant B} f(x) = f(B) \to 0$ as $B \to \infty$. Also, f is bounded. Therefore,

$$\begin{aligned}\int_{-\infty}^{0} dz\, g_\ell(z) f(z+\lambda) &= \int_{-\infty}^{-\lambda/2} dz\, g_\ell(z) f(z+\lambda) + \int_{\lambda/2}^{\lambda} dz\, g_\ell(z-\lambda) f(z) \\ &\leqslant \|f\|_\infty \int_{-\infty}^{-\lambda/2} dz\, g_\ell(z) + \|g_\ell\|_1 \max_{z \geqslant \lambda/2} \{f(z)\}.\end{aligned}$$

As T goes to ∞ so does λ, and both the integral and maximum above tend to zero.

Now consider $\ell = p$. In this case,

$$\begin{aligned}&\Big| \int_{0}^{\infty} dz\, F_T(r,u;\ell,z) F_{T,k}(\ell,z;s,v) \Big| \leqslant C_{q,L} \int_{0}^{\infty} dz\, g_\ell(z) f(z+\lambda) \cdot g_r(u) \cdot e^{\mu(v+\lambda)+\Psi\left(-(v+\lambda)/\Delta_2\right)} \\ &\leqslant \underbrace{C_{q,L}\, \|g_\ell\|_1 \cdot \max_{z \geqslant \lambda} \{f(z)\}}_{\eta_T} \cdot g_r(u) \cdot e^{\mu(v+\lambda)+\Psi\left(-(v+\lambda)/\Delta_2\right)}.\end{aligned}$$

We see that this is small as required. ∎

Lemma 5.11 implies that the matrices $L \cdot B_2$ are small where L is any one of the good matrices mentioned in the opening of this section. So this concludes the proof of Proposition 5.3.

Proof that the KPZ-scaling limit is a consistent family of probability distributions. Let $P(\xi_1,\ldots,\xi_p)$ denote the limiting expression from Theorem 1 as a function of the parameters ξ_k. Namely, recalling $\mathbf{H}_T$ from (1.4),

$$P(\xi_1,\ldots,\xi_p) = \lim_{T\to\infty} \mathbf{Pr}\left[\mathbf{H}_T(x_1,t_1) < \xi_1,\ldots,\mathbf{H}_T(x_p,t_p) < \xi_p\right].$$

From the discussion for the single time law we know that $P(\xi_1) = F_{GUE}(\xi_1 + x_1^2)$, which is a probability distribution in ξ_1 (see [24, 39]). Assume that $p \geqslant 2$. We need to establish that P has appropriate limit values as any $\xi_k \to \pm\infty$ since the other necessary properties are retained in the limit. Consider the parameter ξ_1 for concreteness. Since P is the limit of probability distribution functions,

$$P(\xi_1,\ldots,\xi_p) \leqslant P(\xi_1) = F_{GUE}(\xi_1 + x_1^2).$$

So as $\xi_1 \to -\infty$, $P(\xi_1,\ldots,\xi_p)$ tends to 0 as required.

Now consider the limit as $\xi_1 \to \infty$. We have

$$\mathbf{Pr}\left[\mathbf{H}_T(x_1,t_1) < \xi_1, \mathbf{H}_T(x_2,t_2) < \xi_2,\ldots,\mathbf{H}_T(x_p,t_p) < \xi_p\right] = \mathbf{Pr}\left[\mathbf{H}_T(x_2,t_2) < \xi_2,\ldots,\mathbf{H}_T(x_p,t_p) < \xi_p\right]$$
$$- \mathbf{Pr}\left[\mathbf{H}_T(x_1,t_1) \geqslant \xi_1, \mathbf{H}_T(x_2,t_2) < \xi_2,\ldots,\mathbf{H}_T(x_p,t_p) < \xi_p\right].$$

Since the first two terms above have limits, so does the third, and we find that

$$P(\xi_1,\xi_2,\ldots,\xi_p) = P(\xi_2,\ldots,\xi_p) - \bar{P}(\xi_1,\xi_2,\ldots,\xi_p),$$

where $\bar{P}$ is the limit of the third term. Moreover, $\bar{P}(\xi_1,\ldots,\xi_p) \leqslant 1 - F_{GUE}(\xi_1 + x_1^2)$ since the corresponding pre-limit inequality holds. It follows that $P(\xi_1,\ldots,\xi_p)$ tends to $P(\xi_2,\ldots,\xi_p)$ as $\xi_1 \to \infty$. This shows that the KPZ-scaling limit provides a consistent family of probability distribution functions. It also concludes the proof of Theorem 1.

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